A Historical Approach to Teaching Sequences and Series

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Abstract

The goal of this project is to explore the historical development of the mathematical ideas that underlie the typical curriculum of a high school Algebra 2 class on sequences and series. Then the perspectives gained from the historical research are considered for their potential in developing a collection of curriculum pieces. These curriculum pieces cover a range of difficulty levels, from the foundations of the material up through the concepts that are traditionally more difficult for my students. The pedagogical objectives of these curriculum pieces include having the students: master the appropriate use of sequence notation, gain facility with describing patterns, develop an ability to explain limiting values, gain familiarity with the formulas for Arithmetic and Geometric Series, and understand the methods used to find sums of Arithmetic and Geometric Series. The embedded concepts in history that relate to my objectives are the paradoxes and progress that led mathematicians to clarify their understanding of limits, infinite sums, and their development. The historical aspects of this project find their way into my presentation of all these topics in my classroom.

Introduction

> "Any Civilization worthy of the appellation has sought truth. Thoughtful people cannot but try to understand the variety of natural phenomena, to solve the mystery of how human beings came to dwell on this earth, to discern what purpose life should serve, and to discover human destiny. In all early civilizations but one the answers to these questions were given by religious leaders, answers that were generally accepted. The ancient Greek civilization is the exception. What the Greeks discovered – the greatest discovery made by man – is the power of reason." (Kline, p.9)

When I began working through the historical aspects of this project, I came to realize that my joy within math centers on the use of reason to overcome adversity. Adversity is abundant in the history of limits. There were many paradoxes and puzzles that early mathematicians faced when they began to reason about limits, and many of these puzzles seem worthwhile for my students to consider as well. The two main goals of my MST project have been, first, to challenge myself personally with the historical and mathematical aspects of sequences and series and, second, to make the curriculum component of this project something that I can utilize in my classroom. In particular, my aim with this curriculum is to enhance both my students' conceptual understanding and their appreciation of the subject through the use of a historical perspective.

Prior to beginning this project, I had never studied the early mathematicians – the main characters in the history of sequences and series – nor had I studied the mathematics behind their work. But as a result of this research, I feel I have gained an improved personal understanding of this topic, an understanding will help me to communicate better with my students as they work through the foundations of the subject. One area that I have focused on is the summation of infinite series, as I know that this will directly help me when my students encounter this topic in Algebra 2. Another area of research (one that I neglected at first) was the more basic concepts of limits and limiting values.

Early in the algebra curriculum, students have already studied limits informally through the ideas of asymptotes in radical, rational, exponential, and logarithmic functions. I was led to include this area in my research because in my reading, it became clear to me that these ideas are key to the conceptual understanding of sums and patterns in Algebra 2.

In writing up this project, it has been my aim to work through many of the ideas in the development of series and sequences from a historical perspective, using instructive anecdotes wherever possible. There are two main sections to this project. The first is the result of my mathematical and historical investigations. It consists of a collection of selected episodes in the history of series. The topics in this section include limits, misconceptions, and infinite series. This is followed, in the second section of the project, by the specific curriculum activities that I intend to use in my own classroom to better develop my students' understanding.

Finally, in addition to the two main sections described above, there is one other organizational component of my project. As I started this project, I read a number of biographical accounts of mathematicians who interested me and who contributed to our knowledge of series summation. Accordingly, I have included an appendix that reflects this research and contains a few interesting facts about the Bernoulli brothers and Leonhard Euler. Although this information was quite interesting to me, it was not vital to the actual project; therefore, I have opted to include it as an appendix rather than try to incorporate it all into the main body of the paper.

Selected Episodes in the History of Series and Sequences

Limit Concepts

The first concept we encounter in our study of infinite series is the idea of a limit of a sequence. An easy example to begin our considerations is given by the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and whose n^{th} term is $a_n = \frac{1}{n}$. Notice that, as n gets really large, these terms take the form of the reciprocal of a really large number. This leads us to a seemingly obvious mathematical fact: The values of the terms of this sequence are getting closer and closer to zero. The elegant step we might take for granted is when we try to describe what we mean more exactly, saying, for example that "as n approaches infinity, the limit of $\frac{1}{n}$ is zero." This step represents a rather large transition in thought: going from using numbers in an obvious, quantitative way to using concepts involving the behavior of infinite processes. Today, many of us take this step on a regular basis and yet we might not realize how important it is. These musings lead us quickly into the following discussion of how this conceptualization and attempts at precision in dealing with limits came to be over the course of history.

There are a number of reasons why the history of the concept of limit should be considered important to the Algebra 2 curriculum materials that are developed later. First and foremost is the fact that without the ideas of limits, we would not be able to make precise the idea of finding infinite sums. This is very close to the central topics of our standard curriculum. In fact, the majority of the Algebra 2 curriculum is held together by the common theme of finding patterns and sums. It could be argued that an assumption is being made that the students already know the general ideas about limits. Although my high school students will not get the opportunity to formally study limits until Pre-Calculus or even Calculus, nevertheless, when they explore the curriculum developed in this project, they will encounter many ideas behind limits that will enrich their understanding of many topics in Algebra 2.

Early Development

The original paradox on limits, one that gave mathematicians problems for centuries, was Zeno's Paradox. Roughly speaking, this paradox says that if a man keeps going halfway to a wall, he can never reach the wall, because it will take him forever to get there. This takes us directly to the consideration of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$. If you think of the partial sums of this series, they keep getting closer and closer to one. As we will see later, the partial sums are all strictly less than 1, but converge to 1 in the limit. In this problem, reaching one corresponds to completing the journey – reaching the wall. Of course the partial sums never actually *equal* one. That is where we are first faced with the idea of a limit. We may be tempted offhandedly just to *define* this infinite sum to be equal to one, since we have the advantage of a strong intuition in this problem that precedes a well-formed idea of a limit. But for the moment, we only want to observe that the notion of a limit is central to the resolution of the paradox. Indeed, it is precisely the idea of a limit that allows us to get beyond the time issues that make this puzzle perplexing. We must observe that our time intervals can also be infinitely small so we can "reach" the wall in less than infinite time (Berlinski, p.123). Zeno of Elea who posed the paradox, was alive in the 5^{th} century B.C. so the Greeks had not had the opportunity yet to develop precisely the necessary concepts of limit that let us sum the geometric infinite series in question (Kline p. 349). We will discuss further the geometric infinite series in a later section.

Another of the earliest contributors to the concept of the limit was Antiphon the Sophist (ca 430 B.C.). According to Antiphon, you could analyze the area of a circle by repeatedly doubling the number of sides of a regular polygon inscribed in that circle. Then at each step, the area of the regular polygon would provide you with a better approximation of the area of the circle (see figure 1). This begins to concern the idea of limits as the number of sides becomes indefinitely larger.

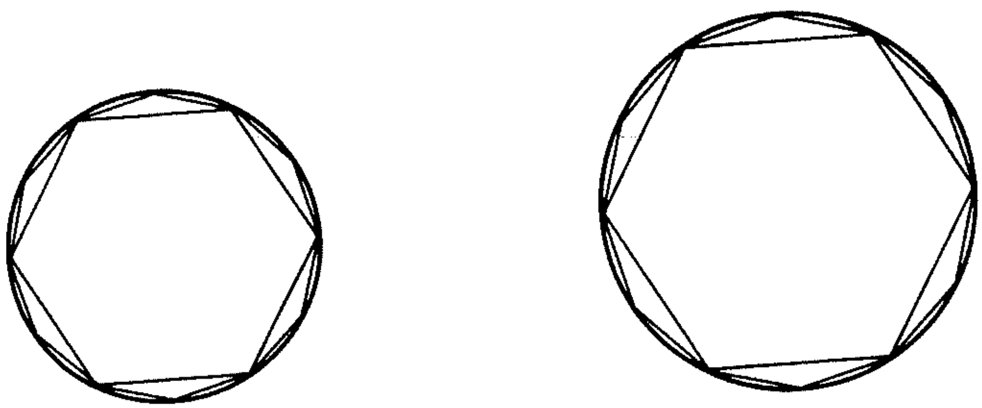


FIGURE 1

(Boyer, p. 101)

This is an example of what the Greeks referred to as the Method of Exhaustion. As you can well imagine, this Greek Method of Exhaustion is directly what led us, in Western civilization through the centuries, to the limit concepts we use today. The figure above shows the six and twelve-sided regular polygons inscribed in a circle (Eves, p.390). Observe the improvement in the approximation when the number of sides increase.

Although Zeno and Antiphon raised these issues, Archimedes was the first person to use the summing of infinite series as a means to calculate finite quantities in his mathematical investigations. It is interesting to note that Archimedes never mentioned infinity, as he preferred to discuss the concept rather obliquely, always with reference to the Method of Exhaustion. The Method of Exhaustion is a quantitative method related to the idea of Antiphon. One finds the area of a shape by inscribing it inside a sequence of polygons whose areas converge to the area of the containing shape. For example, at one point, Archimedes was trying to approximate the circumference of a circle given a diameter of one unit. As we know using our familiar formula for circumference, the circumference comes out to be π. Archimedes' approach was to further develop Antiphon's idea of inscribing regular polygons in the circle to approximate the circumference. Archimedes both inscribed and circumscribed a circle with regular polygons as you can see in figure 2. The circumscribed regular polygon would be a little bit too long and the inscribed regular polygon would be a little bit too short.

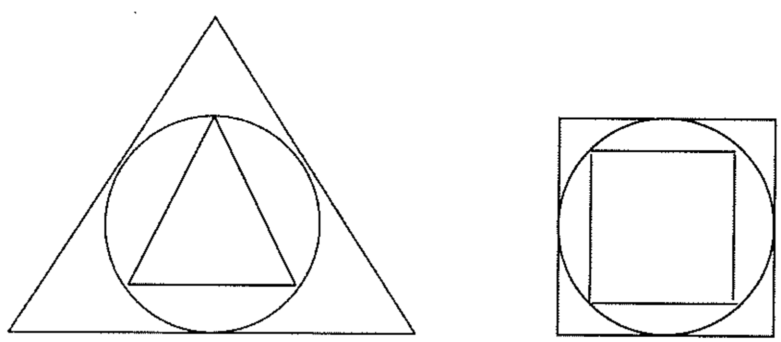


FIGURE 2

He took this process out to the step where the perimeter of the circle was bound inside and out by a pair of 96-sided regular polygons. This led him to the approximation

$$3\frac{284\frac{1}{4}}{2017\frac{1}{4}} < \pi < 3\frac{667\frac{1}{2}}{4673\frac{1}{2}}.$$

From this value for π, one can show that he came within only about $\frac{6}{10000}$ of what we know to be the actual value today. This is quite amazing, as this was in the year 200 B. C.. This shows how, in Ancient Greece, there was a growing realization that, in some circumstances at least, the sum of infinitely many numbers could have finite sums (Struik, p. 50-51). In his book, *Journey Through Genius*, Dunham remarks how amazing it was to accomplish these feats with the ancient Greek methodology, saying, "These computations were the arithmetical counterpart of running the high hurdles wearing ball and chain. Yet by marshaling his enormous intellect and perseverance, Archimedes succeeded in giving the first scientific estimate of the critical constant π" (Dunham, p, 43).

Let us now examine a second way by which Archimedes arrived at an estimate for π – this time using area. He began by considering the area of regular polygons inscribed in a circle. Suppose a regular polygon has n sides each of length b. Make lines from the center of the regular polygon to each vertex of the polygon. The diagram below (figure 3) illustrates the three- and four-sided regular polygon inscribed in a circle. Notice that as the number of sides of the polygon increases, the area of the inscribed regular polygon becomes closer to that of the circle.

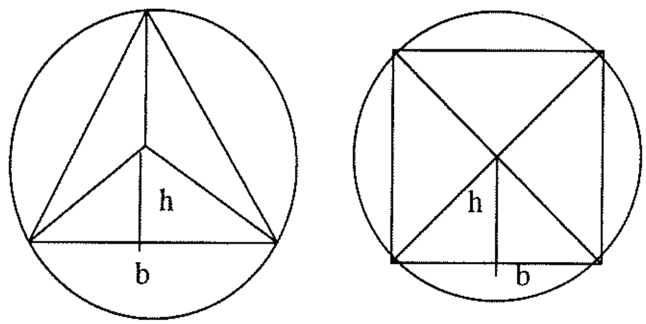


FIGURE 3

Observe that this construction, at each stage, creates a number of triangles, that when summed together, would equal the area of the regular polygon. Let us denote the perimeter of the polygon by Q and, by h, the height of each of the triangles. The great thing about this construction is that now, because of the approach we have taken, we can easily add up the areas of the triangles, as they are all congruent by symmetry! Doing so, we find that

Area of Regular n-sided Polygon

$$= \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + ... + \frac{1}{2}bh, \text{ where the sum contains n terms}$$

$$= \frac{1}{2}h(b + b + b + ... + b) = \frac{1}{2}h(Q)$$

What we have called h is sometimes called the *apothem,* and the above computation has given us the formula that we still use today for computing the area of regular polygons (Dunham, p. 90).

Archimedes then combined this formula for the area of polygons with his intuition for limits to deduce information about the area of a circle. Following him, we begin again with a square that is inscribed inside a circle as shown in figure 4. Bisecting each side of the square shows you how to find the vertices of the 8-sided regular polygon.

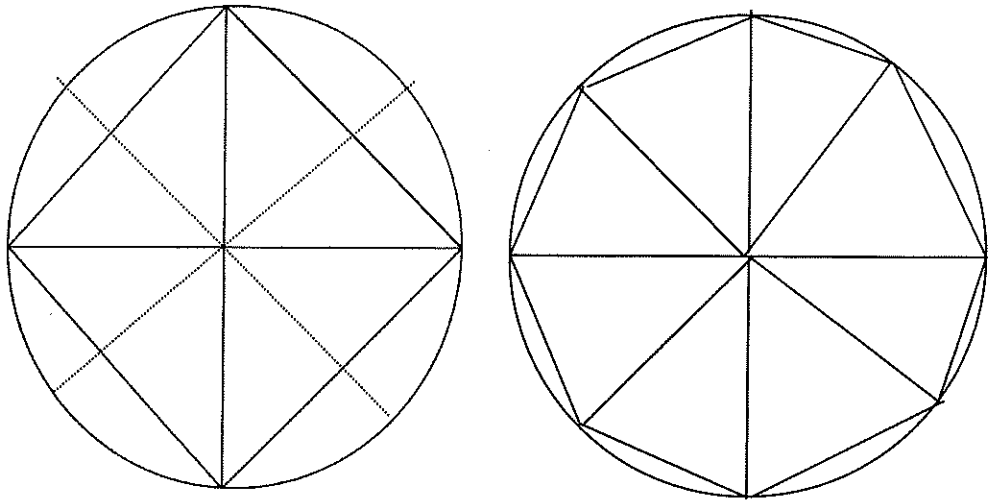


FIGURE 4

Notice that the area of each regular polygon must be less than the area of the circle that they are inscribed in. If we then bisect the edges of this regular octagon, we can then find an inscribed 16-sided regular polygon. The area of each inscribed regular polygon will still be less than the area of the circle. Archimedes used the same idea to circumscribe the circle with regular polygons. This process of doubling the sides of regular polygons, applied to both the inscribed and the circumscribed polygons, led him to a numerical value that, in the limit, approaches the area of the circle. The method of exhaustion is the approach that Archimedes used to analyze the difference of the area of the regular polygon from the area of the circle. The idea is simple: given any small value, one can then always find a regular polygon to make the difference be smaller than that value. Notice the similarity of this to the formal definition of limit we use today. Although it is not expressed in terms of "epsilons" and "deltas," it nonetheless is getting at the same idea: *by continuing the process, we can reduce the error.* In the modern terminology, we are comfortable saying that the error tends to zero (in the limit), whereas for Archimedes, there was always still some error (Dunham, p. 99).

The Chinese and Hindu cultures also approximated the value of π. Tsu Ch'ung-chih came up with an estimate of $\frac{355}{113} = 3.14159292...$ in 480 A.D. Tsu Ch'ung-chih actually used the same approach as Archimedes to find his estimate (Infoplease, p. 2). Then the Hindu mathematician Bharskara came up with the estimate $\frac{3927}{1250} = 3.1416$ around 1185 A. D. Curiously, his approximation was based on the square root of ten, which is close to the value of π. Next, a French mathematician, Francis Viete, used Archimedes' idea and simply carried the computations further. In fact, he actually calculated the area of a 393,216-sided regular polygon that produced a approximation for π that is accurate up to 9 decimal places. The next was a Dutch mathematician, Ludolph van Ceulen, who also followed the Archimedean program and just crunched the numbers longer. He spent many years on his computation and got a 35-decimal place approximation. So the history of π is long and interesting, and although it is a very geometric problem, it is also closely tied to the concept of a limit (Dunham, p. 108).

Let us turn now to another computation from antiquity that involved the idea of a limit. Again this approach comes from the hand of Archimedes. The problem involves a parabolic segment, which, in our case, is a portion of a downward opening parabola that has been cut off by a line across the top (see the figure below). To determine the area of a parabolic segment, Archimedes inscribed a triangle of area A in the parabolic segment sharing the same base and vertex (see figure 5). Then, from the legs going up from the original base, he did the same process to create another triangle, and, continuing in this manner, he filled up the parabolic segment. The object is to obtain the total area as the sum of the triangular areas. Interestingly, the resulting series is a geometric progression as is shown below.

$$A(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+...+\frac{1}{4^{n-1}}+\frac{1}{3}\bullet\frac{1}{4^{n-1}})=\frac{4}{3}A$$

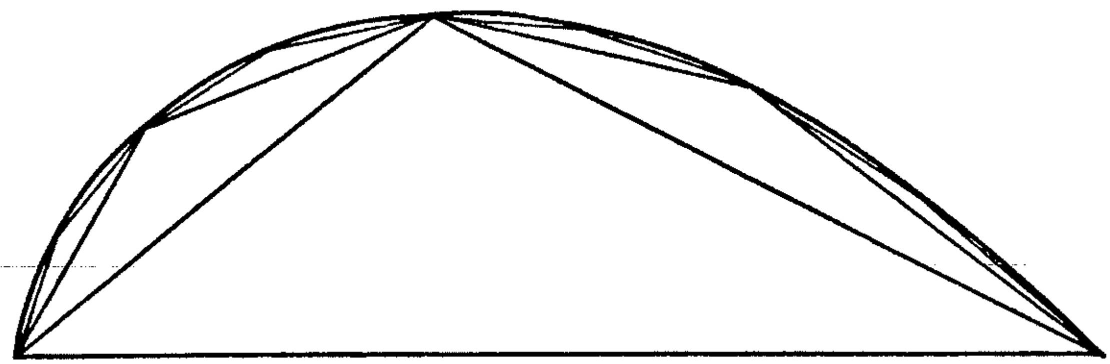


FIGURE 5

(Boyer, 1949, p. 52)

This picture illustrates how Archimedes took more and more triangles from the base of his parabolic segment and, in so doing, approached the area of the entire segment.

Nevertheless, Archimedes did not evaluate the limit in the same manner as we would evaluate it today. Rather, he argued carefully using the Method of Exhaustion, showing that if you gave him any given error, he could always, by taking enough triangles, make the remainder smaller than that error. In today's notation, we'd speak of finding a partial sum that is within ε of the limit. Our notation and algebraic manipulations would have seemed strange indeed to Archimedes, and the steps we follow to evaluate limits were not available to him for summing infinite geometric series.

In 1604 Valerio showed us another example of a limit process: taking an archway shape and inscribing and circumscribing rectangles around it (See figure 6).

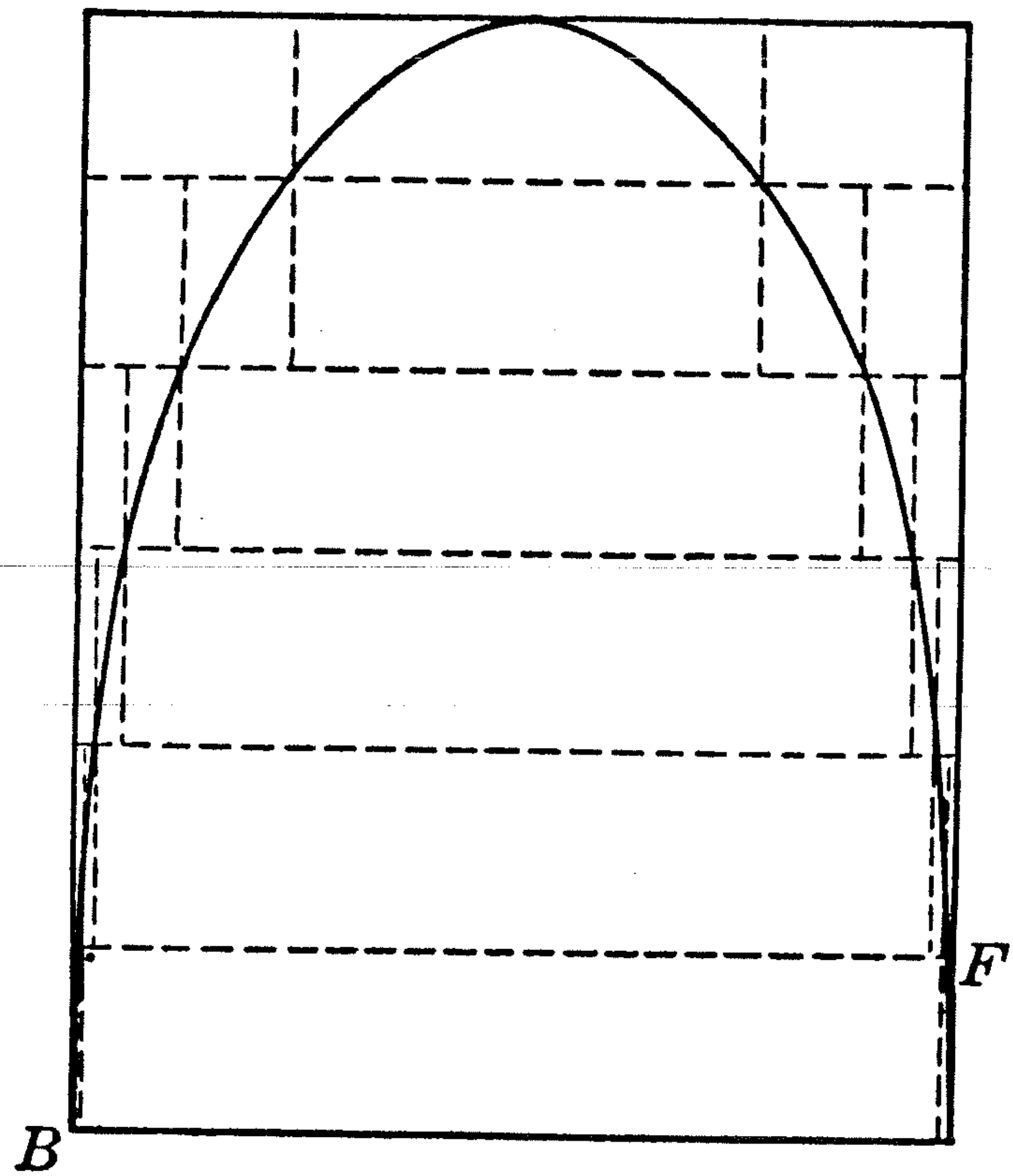


FIGURE 6

(Boyer, 1949, p. 105)

Then Valerio used the Archimedean idea of a limit to argue that one can make the error of the approximation approach zero by squeezing the region in question between the two simpler areas, thereby deducing the area of the arch. It is important to note that Valerio still did not think of this as a formal limit process, with its own algebraic properties, but rather as more of an ad-hoc scheme of approximations that behaves similarly to the Archimedean approximation. Nevertheless, in retrospect, we can clearly connect his approach to this problem with our own modern notion of limit by considering the limit as representing an approximation with infinitely many rectangles – an infinitary 'approximation' that produces zero error (Boyer, 1949, p.105).

Middle Development

John Wallis wrote a book called *Arithmetica Infinitorum* in 1655. He may have dealt with the infinite in a crude manner, but he often was able to find new results. He was interested in computing probabilities of coin flips, and often discovered new ways of looking at familiar formulas. An example of an infinite expression involving π that he found is the following.

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdots}$$

We will consider his derivation below. Although his reasoning in deducing this remarkable identity was not entirely justified, the identity can be made rigorous using calculus techniques we have today. Many of the mathematicians of this time period were also great philosophers, and so the search for new general ideas often led to new ideas in both mathematics and philosophy at the same time (Struik, p. 101-102).

Wallis came up with the above formula for $\frac{\pi}{2}$ in 1655. His development of this formula was based on the area of a circle in a quadrant as shown in figure 7 below.

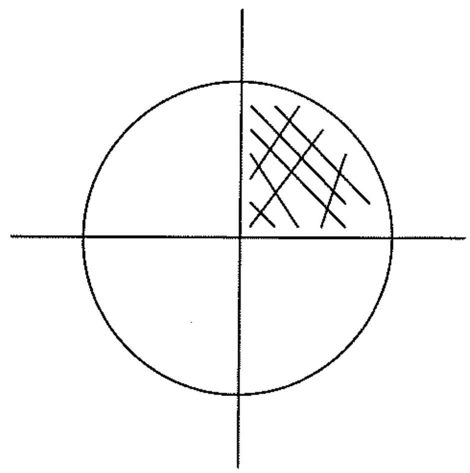


FIGURE 7

He began with the following expression for the relation he wished to establish:

$$\prod_{n=1}^{\infty}\frac{(2n)(2n)}{(2n-1)(2n+1)}=\frac{2}{1}\bullet\frac{2}{3}\bullet\frac{4}{3}\bullet\frac{4}{5}\bullet\frac{6}{5}\bullet\frac{6}{7}\bullet....=\frac{\pi}{2}$$

To prove it, Wallis started with the roots of sine. He used the properties of roots to help him with limits in a way that is short and elegant. For example, the roots of $2x^2-16x+30$ are 3 and 5 so we can write $2x^2-16x+30=2(x-3)(x-5)$. Wallis expressed sine as an infinite product in a similar way. He obtained the following expression, by knowing the 'roots' of sine.

$$\frac{\sin(x)}{x}=k(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})...$$

This expression results from a similar trick one can use to express a polynomial function once the roots are known. The next step was even more interesting, and in modern notation we would describe it as follows. To find the constant k, take the limit of both sides as x goes to zero.

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}k(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})\ldots$$

He knew geometrically that the first limit is 1, which then led him to the conclusion that the constant k must equal 1. Today we of course use L'Hopital's rule and show that $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}$. Nonetheless, he continued as follows. Regrouping,

$$\frac{\sin(x)}{x}=(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})\ldots$$

$$=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\ldots$$

Next he substituted $x=\frac{\pi}{2}$

$$\frac{1}{(\frac{\pi}{2})}=(1-\frac{1}{2^2})(1-\frac{1}{4^2})(1-\frac{1}{6^2})\ldots=\prod_{n=1}^{\infty}(1-\frac{1}{4n^2})$$

Taking reciprocals and simplifying, he concluded that

$$\prod_{n=1}^{\infty}\frac{(2n)(2n)}{(2n-1)(2n+1)}=\frac{2}{1}\bullet\frac{2}{3}\bullet\frac{4}{3}\bullet\frac{4}{5}\bullet\frac{6}{5}\bullet\frac{6}{7}\bullet\ldots=\frac{\pi}{2}$$

$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{4n^2}{(4n^2-1)}$$

And so, by some very ingenious manipulations, Wallis had arrived at his conclusion. The interesting thing to notice, here, is the number of similarities of the method found here in Wallis' work to that of Euler, which we will discuss later (Answers, p. 1-3).

In the 1650's Wallis was the first mathematician to show us a symbol for infinity. His work was a big step towards being able to use a direct arithmetical analysis as opposed to the method of exhaustion (Boyer, 1949, p. 171).

The generally acknowledged founders of calculus, Gottfriend Wilhelm Leibniz and Sir Issac Newton, both had a similar problem in making their computations rigorous. In the notation of Leibniz, this problem was treating $\frac{dy}{dx}$ as a finite quantity. In fact, Leibniz treated his *dy* and *dx* as different things in different situations. He sometimes treated them as finite values, and sometimes as quantities that were less than any assigned value. He did agree, in a letter sent to Foucher in 1693, that he accepted the existence of the infinite to overcome the problems of Zeno's paradox. The works of Newton and Leibniz however, were both based on more fundamental investigations due to Pierre Fermat (Struik, p. 112). We therefore turn our attention to his pioneering work.

Fermat was one of the first mathematicians to merge the concepts of limits and slope. He discovered an approach using slopes that could efficiently locate the maximum and minimum values of many functions. The idea was to find the slope with a close distance between two points and then set the value of the distance equal to zero in the resulting expression. In the notation we use today it would appear as

$$\lim_{E \to 0} \frac{f(x+E) - f(x)}{E}$$

(Boyer, p. 382). You notice that Fermat's error quantity survives in our expression, becaming the E of the above expression, or the 'h' familiar in many modern formulas we use today. Although Fermat did not write his expression in exactly this manner, his basic ideas were the heart of the definition we use today. Boyer words it as follows,

> "Obviously Fermat was not in possession of the limit concept, but otherwise his method of maxima and minima parallels that used in the calculus today, except that now the symbol h or Δx is customarily used in place of Fermat's E. Fermat's process of changing the variable slightly and considering neighboring values has ever since been the essence of infinitesimal analysis." (Boyer, p.383)

Fermat justified the equating of two values at a point by remarking that at a maximum point "they are not really equal but they should be equal" (Boyer, 1949, p. 156).

Modern Development

As calculus developed further, the notion of limit also became further refined. One of the first calculus textbooks in history is remembered today for a limit result it contained that was due to Jean Bernoulli. The result relates derivatives to limits in a novel way. Bernoulli sent the following idea to L'Hospital in a letter:

if f and g are differentiable, $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then

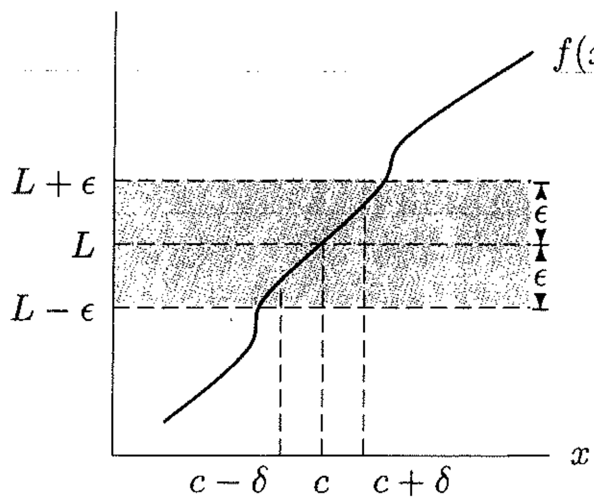
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

After L'Hospital had this result published, it became known as L'Hospital rule. This caused tension, as it was really Jean Bernoulli's discovery. The rule itself is not of central importance to this paper, but the book that it was published in claimed two major concepts. The first was that one could take as equal two quantities, which differ only by an infinitely small quantity. The second was that a curve could be considered as made up of infinitely small straight-line segments that determine, by the angles that they make up with each other, the curvature of the curve. Fermat claimed this earlier and referred to it as pseudo equalities. As you can see, many of the limit ideas they had back around 1700 were expressed in colorful and intuitive language that would still not be regarded as rigorous by today's standards (Boyer, p. 460).

The idea of limits was a crucial part to the development of the modern definition of derivative. Some of the core ideas of calculus were debated for a long time, but there are a few ideas that we keep today and, for our purposes, the fourth one is most important. The ideas are the following:

1. The idea that calculus concerns functions (rather than variables)
2. The choice of the derivative as fundamental concept of the differential calculus (rather than the differential)
3. The concept of the derivative as a function
4. The concept of limit, in particular the limit of a function for explicitly indicated behavior of the independent variable.

(Grattan-Guiness, p. 90)



(Hughes-Hallet, p.128)

As we think about some of the ideas behind the development of limits and limiting values we need to visit the more formal idea of a limit. Given a function $f(x)$, we denote by L the limit at $x = c$. Pick some distance on either side of the limit L. Normally this value is called epsilon. One can always find a delta (a distance on each side of c) so that if an x, distinct from c, is chosen within delta of c, the value of the function will always be within epsilon of L (Berlinski, p.153). Notice that it doesn't matter if the value of the function at c is actually defined. If L is really the limit of f at c, then we must always be able to find that delta that guarantees a value within the distance epsilon.

Another, more advanced aspect of the development of limits concerns the extreme example of function behavior exhibited by fractals. Although we will not develop the theory of fractals in any detail here, we will include a few in the curriculum as a visual illustration of a limit. Therefore, we close this section with a brief review of a few terms. A fractal is a geometric shape which is self-similar and which has a fractal dimension. Visually, if you zoom in as much as you want, you would always see the same pattern over and over again. The Sierpinki Triangle is an example of a fractal. Repeating fractal patterns are really an example of limits. Each step of the pattern is called an iteration. In the curriculum section that follows I have two activities that relate to fractals. This helps reinforce the idea with my students about how the pattern development works. It is great for the students to have a problem where the area is finite and the perimeter is infinite.

Mathematicians who had Misconceptions

Any math teacher has to deal with the fact that students have misconceptions. Sometimes teachers can easily address misconceptions and in other situations students will have them for years and not realize the problem. Perhaps the biggest misconception in a unit on series and sequences is that students often doubt that infinitely many terms can add up to a finite number. As such, it is difficult for them to understand that you can have an infinite series with a finite sum.

To help students get their mind around this issue, we often use the number $\frac{1}{3}$, expressed as

$$\frac{1}{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1,000}+\frac{3}{10,000}+\ldots \qquad \text{(Dunham, p.193)}$$

The terms get negligible quite quickly. This is an illustration that I regularly use with my high school students to try and convince them that infinite series can have finite sums. This was a conceptual difficulty for mathematicians as they struggled to figure out what series had finite sums and which did not.

Instead of looking to the terms heading towards zero, we needed to instead look at the partial sums heading towards a limit of a finite value. The following definition is what Calculus books include as the guide to lead us. Suppose we are given a sequence $S=\sum_{k=1}^{\infty}u_k$. For each natural number n, define

$$s_n=u_1+u_2+\ldots+u_m$$

We refer to s_n as the nth partial sum of S.

Let $\{ s_n \}$ be the sequence of partial sums of the series $u_1 + u_2 + ... + u_k + ...$ If the sequence $\{ s_n \}$ converges to a limit S, then the series is said to converge to S, and S is called the sum of the series. We denote this by writing $S = \sum_{k=1}^{\infty} u_k$ If the sequence of partial sums diverges, then the series is said it diverge. A divergent series has no sum. (Anton, p. 634)

Much of the early development of infinite series was led partially by the Bernoulli brothers. Their notion of convergence was not as formal as what we use today. For example in the 17th century the Bernoulli's were quite comfortable working with series like

$$1 + 2 + 3 + 4 + ...$$

It is clear this series is infinite. We see the idea that it diverges to infinity. The assumption that many students make is an over-generalization of this phenomenon: that any infinite series shouldn't have a sum. This misconception makes studying infinite series challenging.

Mathematicians in the 18th century had similar problems to my students in manipulating infinite processes. They had difficulties manipulating symbols such as $0, \infty, \sqrt{-1}$. Although they of course made many amazing discoveries in their investigations, a number of their results them were later proven to be misconceptions.

One example of this is Euler's equation:

$$1 - 3 + 5 - 7 + ... = 0$$

In modern math we would simply look at the limit of partial sums and see that it doesn't have a limit. In particular, the sequence of partial sums is the following:

$$1, -2, 3, -4, 5, -6, 7, ...$$

Since this sequence of partial sums doesn't have a limit, we conclude that the series doesn't have a sum.

Another example is when Euler takes the following two equations and combines them to the third equation (Struik, p. 124).

$$n + n^2 + n^3 + ... = \frac{n}{1-n}$$

$$and$$

$$1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + ... = \frac{n}{n-1}$$

$$concluded$$

$$... + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + ... = 0$$

This is a combination of two geometric patterns, that appears plausible, and yet is not reasonable. The problem is that the values of n which give convergence in the first series are not the same as those that give convergence in the second series. In particular, the first sum converges for $-1 < n < 1$ and the second for $n > 1 or n < -1$. So no single value of n gives convergence in the third expression. This example illustrates a violation of the domain of inputs of geometric series. The restrictions on the common ratio of a geometric series will be discussed later.

Another example of a mathematician with an important misconception was Guido Grandi, a priest and professor in Pisa. Grandi considered the following: suppose there is a father who bequeaths a gem to his two sons, who each may keep the gem in one-year lengths. Then, he reasoned, it then obviously belongs to each son for one half the time. He modeled this situation with the following formulas.

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \ldots\ldots$$
$$= (1-1) + (1-1) + (1-1) + \ldots\ldots$$
$$= 0 + 0 + 0 + 0 + \ldots.$$

This approach is crazy by modern standards but most people would agree he had a certain logic to his reasoning. The modern mathematical approach is to look at the limit of the sequence of partial sums, which are as follows $1, 0, 1, 0, 1, 0, 1, 0, 1, 0$. Since the partial sums don't have a limit, the series doesn't have a sum. This is a great example of how "everyday" logical reasoning can sometimes lead you down the wrong road mathematically (Struik, p. 125).

As a final example of an important misconception, we turn to Leibniz himself. Leibniz was challenged by his teacher Huygens to determine the following sum.

$$S = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \ldots$$

This series is the sum of the reciprocals of the triangular numbers. Leibniz divided every term by two.

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$$

Then Leibniz replaced $\frac{1}{2}$ with $(1 - \frac{1}{2})$, $\frac{1}{6}$ with$(\frac{1}{2} - \frac{1}{3})$, and $\frac{1}{12}$with $(\frac{1}{3} - \frac{1}{4})$.

$$\frac{1}{2}S = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \ldots$$

Leibniz cancelled the entire right side of the equation to leave him with a 1 on the right.

$$\frac{1}{2}S = 1 \text{ Therefore } S = 2$$

Mathematicians today have reservations about the easy way Leibniz just "canceled" off to infinity. The modern approach to looking at the series is to look at the limit of the partial sums. The partial sums here are $1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, 1, \frac{4}{5}, 1, \frac{5}{6}, 1, \frac{6}{7}, 1, \ldots$.. which do have a limit of one so it wasn't a problem, and the final result he obtained was indeed valid. These examples show us how Grandi and Leibniz both had logical ideas, but, since they did not have the more modern approach of looking at the limits of partial sums, one got it right and one got it wrong (Dunham, p.186).

Arithmetic and Geometric Series

Arithmetic and geometric series are the two main kinds of series we study in the Algebra 2 Curriculum. We will begin with these more elementary concepts and then work our way up through a number of more difficult series. Carl Friedrich Gauss, a mathematician born in 1777, offered quite a major contribution to the curriculum we use in Algebra 2 class today. When Gauss was a young man he was given the problem of adding the numbers from 1-100 to occupy his time. The teacher was quite surprised when he quickly came up with the answer of 5050. He thought of his sum as a number, we shall call it S. Then he wrote terms of the sum backwards.

$$S = 1 + 2 + 3 + 4 + 5 + ... + 98 + 99 + 100$$

$$S = 100 + 99 + 98 + 97 + 96 + ... + 3 + 2 + 1$$

Then he added up the two equations while searching for S

$$2S = 101 + 101 + 101 + 101 + ... + 101 + 101 + 101$$

$$2S = 101(100)$$
$$S = \frac{101(100)}{2}$$
$$S = 5050$$

Gauss' work leads us directly to the formula for arithmetic series we use today in Algebra 2. A sequence or series is said to be <u>arithmetic</u> if the *difference* of consecutive terms is constant. If $a_1, a_2, a_3, ..., a_n$ form an arithmetic sequence and $S = a_1 + a_2 + a_3 + ... + a_n$, then $2S = (a_1 + a_n)n$, so $S = \frac{(a_1 + a_n)n}{2}$. The derivation follows that of the young Gauss, given above. (Dunham, p. 236-238).

After the arithmetic series, the most well known convergent series in math is the geometric series. A sequence or series is said to be <u>geometric</u> if the *ratio* of consecutive terms is constant. As we will see, geometric series can have finite sums even when there are infinitely many terms. For example, consider the series

$$x + x^2 + x^3 + x^4 + \dots$$

with $-1 < x < 1$. The classic approach to prove the sum is $\dfrac{x}{1-x}$ is as follows.

$$S = x + x^2 + x^3 + x^4 + \dots$$

$$xS = x^2 + x^3 + x^4 + x^5 + \dots$$

subtract

$$S - xS = x$$

$$S = \frac{x}{1-x}$$

$$x + x^2 + x^3 + x^4 + \dots = \frac{x}{1-x}$$

An example of this for $x = \dfrac{1}{3}$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{31} + \dots \left(\frac{1}{3}\right)^k + \dots = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

The question comes up why the series converges for x between –1 and 1. But there is a difficulty in the above line of reasoning. The manipulations involved only make sense if you assume the series converges, since we are treating S as a real number. So what the above computations show is that if the series converges, then the sum equals $\dfrac{x}{1-x}$. An obvious example of why the constraint is necessary is to see what it would suggest if $x = 2$.

$$2 + 2^2 + 2^3 + 2^4 + \ldots = \frac{2}{1-2} = -2$$

So a series that would have to diverge to infinity obviously can not add up to a negative number as we are only adding positive numbers (Dunham, 194-195). The easiest way to show the reason that it does converge between –1 and 1 is to look at the partial sum and the limit of that partial sum as you head towards infinity.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{(1-x^n)}{1-x} = \frac{(1-0)}{1-x} = \frac{1}{1-x}$$

This computation shows us how when the value of x, the common ratio is between –1 and 1, the term x^n in the limit goes to zero. Conversely that is the same term that approaches infinity if the value of x isn't between –1 and 1 (Hughes-Hallet, p. 453).

After the last examples, one might think that a series adds to a finite number whenever the numbers in the series approach zero. This is a common misconception among Algebra 2 students that needs to be corrected conceptually. In fact, the converse is true: if a series adds up to a finite number, then it must have terms that approach zero. In the next chapter, we will encounter a series that goes a long ways towards accomplishing this task (Dunham, 194-195).

Harmonic Series

As we continue to study the development of series and sequences, we arrive at another interesting topic; the harmonic series. This topic poignantly raises the issue of which infinite series have finite sums and which ones don't. In the high school setting we discuss lightly the idea that some series have terms whose limit is going to zero but yet fail to have a sum. This is often believed to be impossible, a common misconception of high school students and many college students as well.

The harmonic series is a wonderful example of a series that has terms that head towards zero, and yet has an infinite sum.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots$$

Johann Bernoulli proved the divergence of this series and referred to this example as the "pathological counterexample." He felt it seemed counterintuitive and bizarre. What is so bizarre about the harmonic series is the slow rate of divergence; the sum of the first 83 terms of the series sum barely exceeds 5. For the sum of the Harmonic series to be greater than 20 it takes more than one quarter of a billion terms (Dunham, 195-196).

The proof of the divergence of the harmonic series was published in Jakob Bernoulli's works yet he credited his brother Johann as being the person who came up with the proof. Start with the harmonic series:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots$$

The first step is to get all the numerators to 1, 2, 3, 4, etc. You can notice that series A is the harmonic series without the one.

$$A = \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30} + \ldots$$

Then Johann called the following series C. Then by taking away the next largest term he finds a pattern to use to prove the harmonic series..

$$C = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = 1$$

$$D = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = 1 - \frac{1}{2} = \frac{1}{2}$$

$$E = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

Then by adding the left two columns he gets

$$C + D + E + \ldots = \frac{1}{2} + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{12} + \frac{1}{12} + \frac{1}{12}\right) + \ldots$$

$$= \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \ldots\ldots = A$$

Also by adding the far right column was 1+A so

$$A = 1 + A$$

So if the sum is a finite number A, we have shown it to be also equal to 1+A, this gives us a contradiction, thus A must be infinite (Durham p 197-198).

Johann Bernoulli was actually not the first person in history to prove the divergence of the harmonic series. The first proof was from Nick Oresme (1323 – 1382) a French scholar. Although, apparently mathematicians of Bernoulli's time were not aware of the fact that someone else had already solved the problem. We will now look at Oresme's proof. He first expressed the proof in a verbal way. He said the following:

"… Add to a magnitude of 1 foot: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, foot etc;

the sum on which is infinite. In fact, it is possible to form an infinite number of groups with a sum greater than $\frac{1}{2}$. Thus, $\frac{1}{3}+\frac{1}{4}$ is greater than $\frac{1}{2}$; $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$ is greater than $\frac{1}{2}$; $\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}$ is greater than $\frac{1}{2}$, etc." (Durham, p 202)

To formalize this approach he used numbers from the harmonic series to always find a group of smaller fractions whose sum exceeds $\frac{1}{2}$.

$$1+\frac{1}{2}>\frac{1}{2}+\frac{1}{2}=1$$
$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)>1+\left(\frac{1}{4}+\frac{1}{4}\right)=\frac{3}{2}$$
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)>\frac{3}{2}+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)=\frac{4}{2}$$
$$1+\frac{1}{2}+\ldots+\frac{1}{8}+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)>\frac{4}{2}+\left(\frac{1}{16}+\ldots+\frac{1}{16}\right)=\frac{5}{2}$$

An extension of this is

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}>\frac{k+1}{2}$$

Thus, for any finite number, the harmonic series has enough terms to be greater than the number. The conclusion then is we can always exceed a bigger number, so the series must be infinite (Durham p. 202-203).

There is a third approach using calculus. It is the technique I was taught when I first wanted to look at the divergence of the harmonic series in college. We start with the goal of showing the harmonic series does not converge. The idea is to think of the values of the harmonic series as an approximation to the area under the curve $y=\frac{1}{x}$.

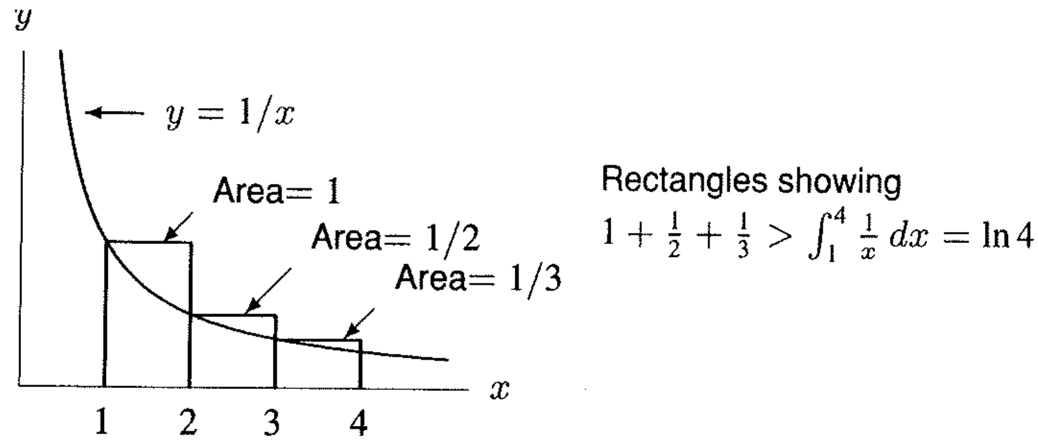


FIGURE 8

(Hughes-Hallett, p. 442)

We need to put the first rectangle with sides 1 and 2 on the x-axis with a height of one. We then continue to make the rectangles one unit wide and with height equal to the corresponding value of the function at the left endpoint of the base as you can see in figure 8. If you compare the area enclosed by the rectangles to the area beneath the curve of $\frac{1}{x}$, the rectangles always have a bigger area.

For example,

$$1+\frac{1}{2}+\frac{1}{3} > \int_1^4 \frac{1}{x}dx = \ln(4)$$

Given the first three rectangles, we can see that the partial sum is greater than ln(4).

Then we can transfer this idea to the nth partial sum, since the function $\frac{1}{x}$ is strictly decreasing on $(0,\infty)$, finding that

$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} > \int_1^{n+1} \frac{1}{x}dx = \ln(n+1)$$

This shows that the nth partial sum is always going to be greater than ln(n+1), suggesting that as n goes to infinity the partial sums exceed ln(n+1), which increases forever. So we conclude that the series does not converge (Hughes-Hallett, p. 442).

A different way to prove harmonic series is to use the Integral Test. It is similar to the last approach but more formal.

$$\sum_{k=1}^{\infty}\frac{1}{k} \text{ diverges since}$$

$$\int_1^{\infty}\frac{1}{k}dk = \lim_{a\to\infty}\int_1^{a}\frac{1}{k}dk$$

$$\lim_{a\to\infty}[\ln k]_1^a = \lim_{a\to\infty}(\ln a - \ln 1)$$

$$\lim_{a\to\infty}(\ln a - 0) = \lim_{a\to\infty}(\ln a) = \infty$$

The Integral Test of course states a series with positive terms that is also written as a function that is decreasing and continuous from point a to infinity. Then the following two expressions converge or both diverge.

$$\sum_{k=1}^{\infty}u_k \text{ and } \int_1^{\infty}f(x)dx$$

It is interesting to note that the Bernoulli's technique uses a contradiction to show the sum must be infinite. Even though the last two techniques use different ideas they take a similar approach: to show the sum is always greater than something else that is known to diverge to infinity, thus the sum must also diverge to infinity. It is also interesting to see how efficiently we can use calculus in the third technique; we actually accomplish a quick integral test to deduce that the series doesn't converge.

Euler: The Reciprocals of the squares

The sum of reciprocals of the squares were a natural next challenge in the progression of how people learned to understand the series in mathematics. Jakob Bernoulli tried to solve this series and then turned his attention to another series when he could not figure it out. His brother Johann figured it out partially, but it was Euler, his pupil, who would eventually solve the problem. The following is the initial work by Johann.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{16^2}+\ldots=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$$

He compared these values to the triangle numbers. The triangular numbers are 1, 3,6,10,15,..... Notice that if you add a new row of dots to the bottom, which is like adding the next larger integer (see Figure 9). He noticed

$$\frac{1}{4}<\frac{1}{3},\frac{1}{9}<\frac{1}{6},\frac{1}{16}<\frac{1}{10}$$

and the reciprocals of the squares are less than the triangular numbers. Stated generally, his observation was that

$$\frac{1}{k^2}<\frac{1}{\frac{k(k+1)}{2}}$$

The grid below in Figure 9 shows you the square numbers (down the left side) and the triangular numbers (down the right side). As you can see, the squares are a larger quantity than the triangular numbers at the each stage. That is why, when you take the reciprocals, the opposite is true.

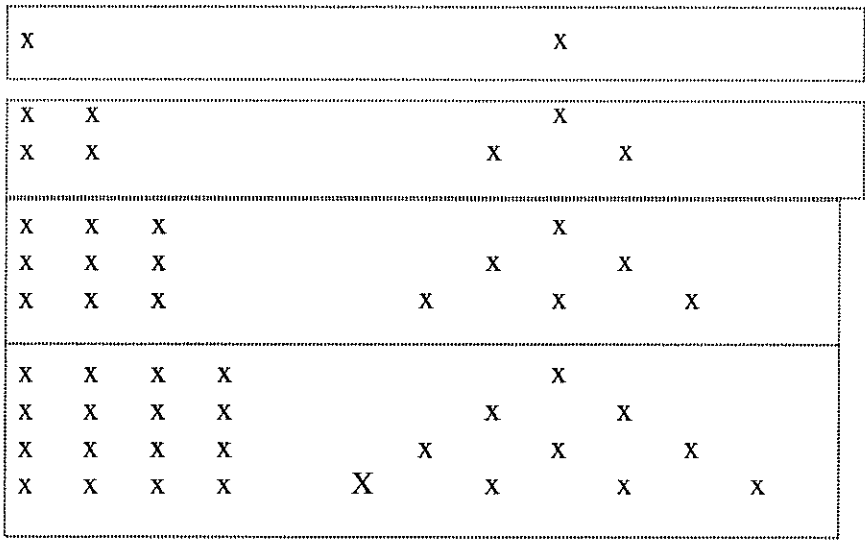


FIGURE 9

Johann Bernoulli's illustration above shows that the k^{th} term of the squared series is always smaller than the k^{th} term of the triangular reciprocals.

$$1+\frac{1}{4}+\frac{1}{9}+...+\frac{1}{k^2}+...<1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+...$$
$$...+\frac{2}{k(k+1)}+...=2(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+...)=2(1)=2$$

This shows the sum must be less than two. This is one of the earliest examples of the idea of a "comparison test" for convergence. But as mentioned above, the question as to what value the series approached wasn't fully solved until 1784, when it was put to rest by Leonhard Euler (Durham p 205-206).

Euler started his attack on this series simply by adding up more and more of its terms. He carried the approximate sum up to 20 decimal places. This was very difficult given the lack of computers. The number looked like it was approximately 1.6449, which didn't look familiar to Euler. Then he writes, "... quite unexpectedly I have found an elegant formula depending on π" (Dunham, p. 212).

To take us through the proof we need to consider a number of ideas that helped Euler develop his approach to solve this problem. It certainly is an approach that the average mathematician of their time and nearly any mathematician of our time would have never considered because of its unique qualities. It is important to keep in mind that many of Euler's manipulations were cavalier, in the sense that he had not proved that all his series were absolutely convergent(which is what one must show before concluding that the terms can be rearranged).

Euler's first idea was to utilize the periodic sine function. In particular, he wanted to utilize the zeros of the sine function. Having infinitely many zeros at integer multiples of π would become essential in his proof. The other aspect of the sine function he used was the infinite series expansion.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

The second important idea in this proof comes from factoring algebra: the idea being that if roots of a polynomial P(x) are a, b, c, and d, and if $P(0) = 1$,

$$P(x) = \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right)\left(1 - \frac{x}{c}\right)\left(1 - \frac{x}{d}\right).$$

This is a slightly different way to think about how to write a polynomial in factored form than we may be use to, but it maintains $P(0) = 1$ and it keeps the correct roots, so the expression must be correct.

Euler decided this principal should be extended to the infinite. He saw the pattern and so he used it with the infinite series he was studying. Using these two main ideas now prepare us to prove the sum of the reciprocals of the squares.

Theorem: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{k^2}+\ldots=\frac{\pi^2}{6}$

Euler started by using the following function

$$f(x)=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots$$

Multiply by $\frac{x}{x}$ which does make us require that $x \neq 0$

$$f(x)=x\left[\frac{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots}{x}\right]$$

$$f(x)=\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\ldots}{x}$$

$$f(x)=\frac{\sin x}{x}$$

We have already said $x \neq 0$ so we are really looking for the zeros of f(x) which is going to be where sinx=0. The zeros are going to be $\pm\pi$,$\pm 2\pi$, $\pm 3\pi$, etc. Applying his factoring trick, Euler reasoned that

$$f(x)=\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{-\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{-2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1-\frac{x}{-3\pi}\right)\ldots$$

$$=\left[\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\right]\left[\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\right]\left[\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\right]$$

Which means

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots=\left[1-\frac{x^2}{\pi^2}\right]\left[1-\frac{x^2}{4\pi^2}\right]\left[1-\frac{x^2}{9\pi^2}\right]\left[1-\frac{x^2}{16\pi^2}\right]\ldots$$

Euler saw that you could multiply out the factored right side of the equation. If you compared coefficients for the constant term, you would get one, and Euler realized that the coefficient of the x^2 term was the one he needed. The higher powers will not matter

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots=1-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots\right)x^2+(\ldots)x^4-\ldots$$

Therefore

$$\frac{-1}{3!}=-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots\right)$$

$$\frac{1}{6}=\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots\right)$$

$$\frac{1}{6}=\frac{1}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\right)$$

$$\frac{\pi^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$$

Euler had shown something mathematicians had been spending decades trying to figure out. It held with Jakob Bernoulli's discovery that the sum was less than two and matched Euler's 1.6449 approximation. What is amazing and hard to believe is the formula for summing up squares is related to π that comes from the relationship of a circle (Dunham, p. 212-217).

Euler found a second technique for finding the sum of the reciprocals of the squares which follows below.

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\left[1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\ldots\right]+\left[\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}+\ldots\right]$$

$$=\left[1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\ldots\right]+\frac{1}{4}\left[1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots\right]$$

$$=\frac{\pi^2}{8}+\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}\quad \text{thus}\quad \frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{8}$$

$$\text{so then}\quad \sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{4}{3}\bullet\frac{\pi^2}{8}=\frac{\pi^2}{6}$$

This second approach Euler found will only work if you know the sum of the reciprocals of the odd square numbers which he did, as we will see in the next chapter. It is quite amazing that he found multiple ways to find the sum of the reciprocals of the squares (Dunham, p. 57).

Euler: More Sums

As a high school teacher, you try to work with your students to find other patterns in series and try to extend what they learn to new contexts. This is something you encourage them to do -- to push themselves. After Euler found the reciprocal of the squares, he didn't stop, he pushed forward and used it to find many other sums and pushed his own understanding.

Euler used his results of summing the reciprocals to find other results. The first is the sum of reciprocals of the even squares

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{100}+\ldots+\frac{1}{(2k)^2}+\ldots$$
$$\frac{1}{4}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots+\frac{1}{k^2}+\ldots\right)$$
$$\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{24}$$

Then Euler could apply the previous two results to find the sum of reciprocals of the odd squares.

Total – Even squares = odd squares

$$\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}$$

Then Euler moved on to a tougher question of the reciprocals of the fourth powers of integers

$$1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{1296}+\ldots+\frac{1}{k^4}+\ldots$$

This took him back to his key equation from proving the squares and trying to figure out what the coefficient of the x^4 term from the infinite multiplication of the factor equation would be.

Euler looked at a pattern of the following

$$\left(1-ax^2\right)\left(1-bx^2\right)$$

and

$$\left(1-ax^2\right)\left(1-bx^2\right)\left(1-cx^2\right) \text{ Etc.}$$

He noticed the coefficient of the x^4 was

$$\frac{1}{2}\left[(a+b+c+d+\ldots)^2-(a^2+b^2+c^2+d^2+\ldots)\right]$$

So he went back to his old favorite equation

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots=\left[1-\frac{x^2}{\pi^2}\right]\left[1-\frac{x^2}{4\pi^2}\right]\left[1-\frac{x^2}{9\pi^2}\right]\left[1-\frac{x^2}{16\pi^2}\right]\ldots$$

He utilized his pattern where $a=\dfrac{1}{\pi^2}$, $b=\dfrac{1}{4\pi^2},\ldots$

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots=1-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots\right)x^2+$$

$$\frac{1}{2}\left[\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots\right)^2-\left(\frac{1}{\pi^4}+\frac{1}{16\pi^4}+\frac{1}{81\pi^4}+\frac{1}{256\pi^4}+\ldots\right)\right]x^4$$

Now going back to the idea of setting coefficients equal to each other

$$\frac{1}{5!}=\frac{1}{2}\left[\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots\right)^2-\left(\frac{1}{\pi^4}+\frac{1}{16\pi^4}+\frac{1}{81\pi^4}+\frac{1}{256\pi^4}+\ldots\right)\right]$$

$$\frac{1}{120}=\frac{1}{2}\left[\frac{1}{\pi^4}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\right)^2-\frac{1}{\pi^4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\right)\right]$$

$$\frac{1}{120}=\frac{1}{2\pi^4}\left[\left(\frac{\pi^2}{6}\right)^2-\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\right)\right]$$

$$\frac{1}{120}=\frac{1}{72}-\frac{1}{2\pi^4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\right)$$

$$\frac{1}{2\pi^4}\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots\right)=\frac{1}{72}-\frac{1}{120}=\frac{1}{180}$$

$$1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots=\frac{1}{180}\bullet\frac{2\pi^4}{1}=\frac{\pi^4}{90}$$

What is scary is that Euler continued and did the reciprocals of the 6^{th} powers.

$$1+\frac{1}{64}+\frac{1}{729}+\frac{1}{4096}+\ldots+\frac{1}{k^6}+\ldots=\frac{\pi^6}{945}$$

He continued doing this for all the even powers up to the 26^{th} power.

$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\ldots+\frac{1}{k^{26}}=\frac{1315862}{11094481976030578125}\pi^{26}$$

With all these patterns of the reciprocals of even powers, what about the reciprocals of the odd powers?

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\ldots$$

In the last couple hundred years no one has figured out this sum. The only conjecture is that it equals $\frac{p}{q}\pi^3$ for some integers p and q. No one has been even able to determine whether this conjecture is true. This shows what an amazing discovery it was that Euler made during his life (Dunham p. 219-222).

Euler was a man who found some things that we could go so far as to call a little bit crazy. He took a number of infinite series that no one would ever think of and found their sums. Take the following infinite series $\frac{1}{15}+\frac{1}{63}+\frac{1}{80}+\frac{1}{255}+\frac{1}{624}+.....$, in which most people wouldn't even find a pattern. The pattern is that the denominators are each one less than the perfect squares that can be written as other integer powers. Euler used his haphazard approach of manipulating sums to find the sum to be $\frac{7}{4}+\frac{\pi^2}{6}$. (Dunham, p.67) My hope is to convey some of this joy of discovery with my students through the activities that I have developed in the following curriculum section.

Algebra 2 Series and Sequences & Curriculum

Intro to Curriculum and Relation to Research

As I researched this topic my ultimate goal was to create materials I could use in my Algebra 2 class to help the students better understand the ideas involved in sequences and series. The following are the objectives in Algebra 2 Sequences and Series.

1. Use sequence notation
2. Describing patterns in sequences
3. Explain limiting values (non-formal)
4. Write explicit formulas for Arithmetic Series
5. Write explicit formulas for Geometric Series
6. Find sums of Arithmetic Series
7. Find sums of Geometric Series

The materials are not a complete unit and would need to be appropriately supplemented. The goal of these materials is to give teachers activities that can help their students explore the ideas of sequences and series. I also had three students who offered to assist me and try out the activities to help me critique them before I use them in mass production. The three students shall be referred to as student number one, two, and three. First, I allowed them to work on each activity as a small group. Then I sat down and reviewed it with them together. I was interested in getting a complete an understanding of their experience as I could. The students seemed to become much more comfortable making constructive criticism after the first two activities. They realized I was trying to get their input to make this better for next year when I would use them with an entire class. I wrote all of the activities and I give full permission for teachers to use them in their classroom if they so desire.

All of the research I have done was geared toward creating new curriculum that can be used in my classroom. I began with the curriculum ideas and then went in search of the "math that made them work". In my experience, I have often found that some of the formulas the textbooks use for arithmetic and geometric series have been lacking in some respects. It would be desirable, then to have better materials to use with this unit in the book to accomplish the objectives I listed above. This is why I approached the main ideas of the historical development of limits and sums of series and the math behind them. The better I can understand the development of limits and sums of series, the better I believe I can help my students understand the objectives listed above.

This is the same reason I studied misconceptions in series and sequences. With a deeper understanding of my own misconceptions, it will perhaps be easier to help my students through their misconceptions. These are the reasons that led me to the research I have collected in this paper.

As I picked out the topics for my activities, I chose to start with foundational ideas that I believe will most help my students to later comprehend the tougher concepts of limits and sums. The first activity, concerning notation, is very straightforward and foundational for my students. The second and third activities are intended to relate the arithmetic and geometric formulas. My high school feels this is the conceptual way to present it, and yet no book we have found does it this way. The fourth activity I created to spur a discussion about how sometimes infinite series have sums and sometimes they do not. Since my high school students usually are dealing with arithmetic and geometric series, it is important not to plant a seed of misconception that all series whose terms have a limit of zero have a sum. Then the final two activities are examples of fractal patterns that incorporate limits as sums of areas and perimeters.

Activity 1: Introduction to Sequences and Series

Objective:

The Student will be able to (TSWBAT):

1. describe patterns in sequences to show they see how they were derived.
2. use rules to write out the first few terms of the sequence.
3. write rules of sequences using the patterns they have seen.

Sequence of Activities & Prior Knowledge:

This activity would be used after a short introduction of sequences. The students need to know the notation of sequences so they understand the a_n notation.

Student Work Concerns:

It was interesting to see student number three describe things based on the formula. On the part related to making formulas they went further than most classes would be able to due to the students' prior experience. The number 17 problem gave them difficulty as they saw the absolute value arithmetic pattern with the alternating sign. Student three eventually found a formula that would work, but that could be written in a more refined nature. It was a lot of fun to watch these students struggle while having fun exploring the options to create the alternating signs.

Teachers' notes:

The writing of formulas is less important at this point than the ability to describe patterns. On the first part, writing out the sequences, the students did a nice job of writing out the patterns and describing them.

In the Curriculum section you have the key, the updated worksheet, and it follows with an example of student work.

Introduction to Sequences and Series

Name________________

Date ________________

Write out the first six terms of the sequence. Describe the pattern of the numbers you see in the sequence.

1. $a_n = 2n + 3$

2. $a_n = 2^n + 1$

3. $a_n = -3n + 6$

4. $a_n = (-2)^n$

5. $a_n = n^3$

6. $a_n = n^2$

7. $a_n = n^2 - 1$

8. $a_n = \dfrac{n+1}{n^2}$

Describe the pattern you see in the sequence, write the next three terms, and try to write a rule to explain it.

9. $1, 4, 7, 10, 13, 16, \ldots\ldots$

10. $\frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}, \ldots\ldots$

11. $\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \ldots\ldots$

12. $2, 4, 8, 16, 32, \ldots\ldots$

13. $1, 4, 9, 16, 25, 36, \ldots\ldots$

14. $2, 5, 10, 17, 26, 37, \ldots\ldots$

15. $\frac{-3}{3}, \frac{-2}{5}, \frac{-1}{7}, \frac{0}{9}, \ldots\ldots$

16. $8.6, 7.7, 6.8, 5.9, \ldots\ldots$

17. $2, -4, 6, -8, 10, \ldots\ldots$

18. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots\ldots$

Introduction to Sequences and Series

Name: Student #1 [redacted]
Date: June 3, 2005

Write out the first six terms of the sequence. Describe the pattern of the numbers you see in the sequence.

1. $a_n = 2n + 3$

n	1	2	3	4	5	6
a_n	5	7	9	11	13	15

an is going up by 2.

2. $a_n = 2^n + 1$

n	1	2	3	4	5	6
a_n	3	5	9	17	33	65

2·2·2·2=16·2=32

well, they're doubling −1

3. $a_n = -3n + 6$

n	1	2	3	4	5	6
a_n	3	0	-3	-6	-9	-12

starting from 3 and −3 each time.

4. $a_n = (-2)^n$

n	1	2	3	4	5	6
a_n	-2	4	-8	16	-32	64

even powers = positive #'s
odd powers = negative #'s
The # alone (w/o the +/-) doubles each

5. $a_n = n^3$

n	1	2	3	4	5	6
a_n	1	8	27	64	125	216

The #'s (n) are getting tripled by the n.

6. $a_n = n^2$

n	1	2	3	4	5	6
a_n	1	4	9	16	25	36

+3 +5 +7 +9 +11

The #'s an are going up by odd #'s +3, +5, +7, +9 ... etc.!

7. $a_n = n^2 - 1$

n	1	2	3	4	5	6
a_n	0	3	8	15	24	35

+3 +5 +7 +9 +11

The #'s are going up by odd #'s +3, +5, +7, +9, +11

8. $a_n = \frac{n+1}{n^2}$

n	1	2	3	4	5	6
a_n	2	3/4	4/9	5/16	6/25	7/36

72/36, 27/36, 16/36, 11.25/36, 8.64/36, 7/36

well the fractions are getting smaller and the numerators go up by 1 and the denominators go up by odd #'s

Describe the pattern you see in the sequence, write the next three terms, and try to write a rule to explain it.

9. 1,4,7,10,13,16,......

goes up by 3

$a_n = 3n - 2$

n	7	8	9
a_n	19	22	25

10. $\frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}$,......

top # goes up by 1 and bottom # goes up by 1 but starting at different points.

$a_n = \frac{n+3}{n+5}$

$\frac{9}{11}, \frac{10}{12}, \frac{11}{13}$

11. $\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \ldots$

Numerator goes up by 1
Denominator goes up in increments of 5

$a_n = \frac{n}{5n}$ $\frac{4}{20}, \frac{5}{25}, \frac{6}{30}$

12. 2, 4, 8, 16, 32, ...

$a_n = n^2$ doubles each time.
64, 128, 256

13. 1, 4, 9, 16, 25, 36, ...

$a_n = n^2$

Going up by odd #s

49, 64, 81

14. 2, 5, 10, 17, 26, 37, ...

50, 65, 82

$a_n = n^2 + 1$

going up by odd #s starting at 2

15. $\frac{-3}{3}, \frac{-2}{5}, \frac{-1}{7}, \frac{0}{9}, \ldots$

top # goes up by 1
bottom # goes up by 2

$a_n = \frac{n-4}{2n+1}$

$\frac{1}{11}, \frac{2}{13}, \frac{3}{15}$

16. 8.6, 7.7, 6.8, 5.9, ...

−.9 each time

$a_n = 9.5 - .9n$

n	5	6	7
a_n	5	4.1	3.2

17. 2, −4, 6, −8, 10, ...12, 14, −16

going up by evens, but neg every other one

$a_n =$

$a_n = 2n$

I gave up look at this! ←

18. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots$

$a_n = \frac{n}{3}$ numerator goes up each time

$\frac{5}{3}, \frac{6}{3}, \frac{7}{3}$

Introduction to Sequences and Series

Name Key

Date ____________

Write out the first six terms of the sequence. Describe the pattern of the numbers you see in the sequence.

1. $a_n = 2n + 3$

5, 7, 9, 11, 13, 15

Start at 5 go up by 2 each time

2. $a_n = 2^n + 1$

3, 5, 9, 17, 33, 65

one more than the powers of 2

3. $a_n = -3n + 6$

3, 0, ⁻3, ⁻6, ⁻9, ⁻12

Starting at 3 going down by 3 each time

4. $a_n = (-2)^n$

⁻2, 4, ⁻8, 16, ⁻32, 64

Start at ⁻2 and times by ⁻2 each time

5. $a_n = n^3$

1, 8, 27, 64, 125, 216

the numbers are the perfect cubes

6. $a_n = n^2$

1, 4, 9, 16, 25, 36

the numbers are the perfect squares

7. $a_n = n^2 - 1$

0, 3, 8, 15, 24, 35

the numbers are one less than the perfect square list

8. $a_n = \frac{n+1}{n^2}$

$\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, \frac{7}{36}$

The top numbers start at 2 and go up by 1. The bottom number is the perfect squares

Describe the pattern you see in the sequence, write the next three terms, and try to write a rule to explain it.

9. 1,4,7,10,13,16,......

+3 each time

$a_n = 3n + {}^{-}2$

10. $\frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}, \ldots$

Start at $\frac{4}{6}$ and then add one to the top and bottom of the fracti

$a_n = \frac{n+3}{n+5}$

11. $\frac{1}{5},\frac{2}{10},\frac{3}{15},\ldots$

up by one on top
Multiples of 5 on bottom

$$a_n = \frac{n}{5n}$$

12. 2,4,8,16,32,......

times by 2 each time

$$a_n = 1\cdot(2)^n$$

13. 1,4,9,16,25,36,.....

Perfect Squares

$$a_n = n^2$$

14. 2,5,10,17,26,37,......

one more than perfect squar

$$a_n = n^2 + 1$$

15. $\frac{-3}{3},\frac{-2}{5},\frac{-1}{7},\frac{0}{9},\ldots$

On top start at -3 and go up by one.
On bottom start at 3 and go up by 2's

$$a_n = \frac{n-4}{2n+1}$$

16. 8.6,7.7,6.8,5.9,....

start at 8.6 and take away .9 each time

$$a_n = -.9n + 9.5$$

17. 2,−4,6,−8,10,.....

Adding two each time with a change of sign

$$a_n = (2n)(-1)^{n+1}$$

18. $\frac{1}{3},\frac{2}{3},\frac{3}{3},\frac{4}{3},\ldots$

Bottom number stays the same
Top up by one

$$a_n = \frac{n}{3}$$

Activity 2: Arithmetic Sequences and Series

Objective:

TSWBAT:

1. use the linear relationship to write an explicit rule.
2. graph the linear relationship of an arithmetic sequence.
3. find sums of arithmetic series.

Sequence of Activities & Prior Knowledge:

This activity can be used once students have explored the nature patterns in activity 1. The students would have the need to have the knowledge of sequence notation. The students would also need to be familiar with traditional graphing. The other prior knowledge needed is the slope-intercept form of equations.

Student Work Concerns:

I had the students graph the sequence and they very easily saw the relationship to the traditional line equation. I put in a couple of arithmetic sums to see if they could figure out the pattern of arithmetic sums and they did, even though it took them quite a long time as they tried a lot of different approaches. The students had good success with this worksheet and didn't note any changes that would be necessary to use in the future.

Teacher Notes:

All of the second year algebra books I have seen use the formula $a_n = d(n-1) + a_1$ but the teachers at my school strongly feel we should use the formula $a_n = dn + a_0$ instead. In Algebra you have studied lines and by using a_0 and d we can directly relate it to the study of lines and $y = mx + b$. This way it will be a direct relation to the other prior knowledge that they have. The reason that books use the other formula is because there is technically not a a_0 term and use shifting instead.

Arithmetic Sequences and Series

Name__________________
Date ___________________

Graph the following sequences using the term position as the Domain and the values as Range.

1. 3,5,7,9,11,…….

2. 8,5,2,-1,……..

3. 3,7,11,15,……..

4. $\frac{7}{2},3,\frac{5}{2},2,\frac{3}{2},\ldots\ldots$

On the above graphs, if the Domain were all real numbers what would the equation of a sketch going through the points look like and what would the equation of the graph be.

1.

2.

3.

4.

As you can see the equation for this type of graph is linear when you keep adding the same amount. The traditional equation for dealing with this kind of equation is

$$y = mx + b$$

We now want to change this to Sequences notation so we switch the y to a_n and x to n to tell us we only want integer values for n.

$$a_n = (?)n + (?)$$

Together with your table partner please try and describe how you decide what the m and b change to in sequence notation.

m

b

5. Given the numbers from 1-100 please find the sum of all the numbers.

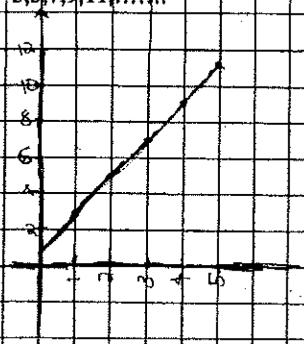
6. Given the numbers from 1-1000 please find the sum of all the numbers. (hint, try to find an easy way to add them up with patterns.

Arithmetic Sequences and Series

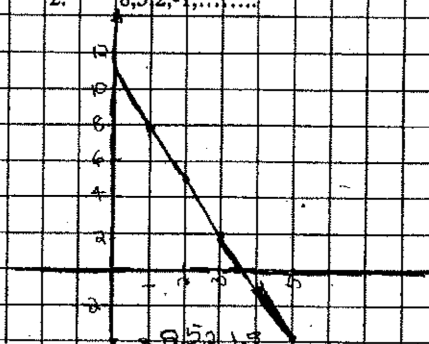
Name: Student #2 [redacted]
Date: 06/06/05

Graph the following sequences using the term position as the Domain and the values as Range.

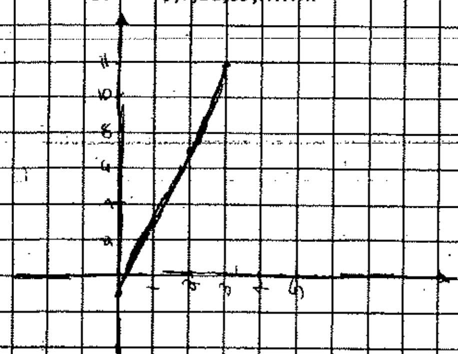
1. 3,5,7,9,11,.......



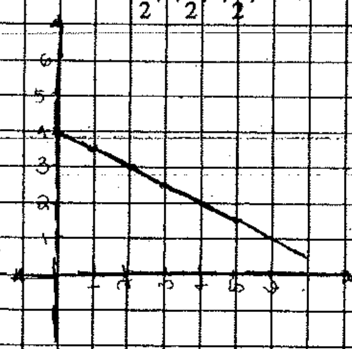
2. 8,5,2,-1,.......



3. 3,7,11,15,.......



4. $\frac{7}{2}, 3, \frac{5}{2}, 2, \frac{3}{2}, \ldots\ldots$



On the above graphs, if the Domain was all real numbers what would the equation of a sketch going through the points look like and what would the equation of the graph be.

1. y = 2x + 1

2. y = -3x + 11

3. y = 4x - 1

4. y = -.5x + 4

As you can see the equation for this type of graph is linear when you keep adding the same amount. The traditional equation for dealing with this kind of equation is

$$y = mx + b$$

We now want to change this to Sequences notation so we switch the y to a_n and x to n to tell us we only want integer values for n.

$$a_n = (?)n + (?)$$

Together with your table partner please try and describe how you decide what the m and b change to in sequence notation. There would be no change b/c sequence notation.

m slope

b y-intercept

5 Given the numbers from 1-100 please find the sum of all the numbers.

1 + 100 = 101
· 50
5050

6 Given the numbers from 1-1000 please find the sum of all the numbers. (hint, try to find an easy way to add them up with patterns.

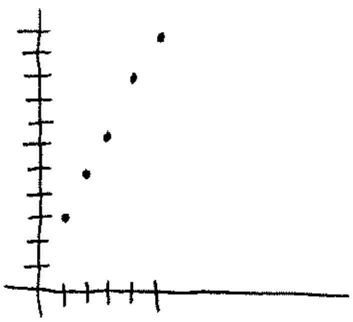
1 + 1000 = 1001
· 500
500500

Arithmetic Sequences and Series

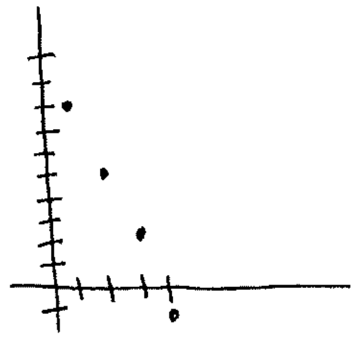
Name KEY
Date ____________

Graph the following sequences using the term position as the Domain and the values as Range.

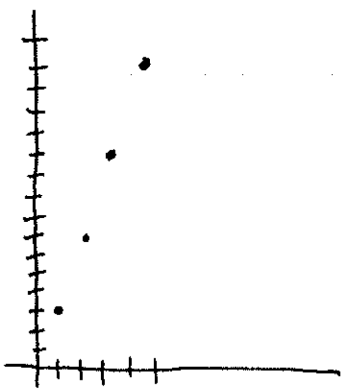
1. 3,5,7,9,11,.......



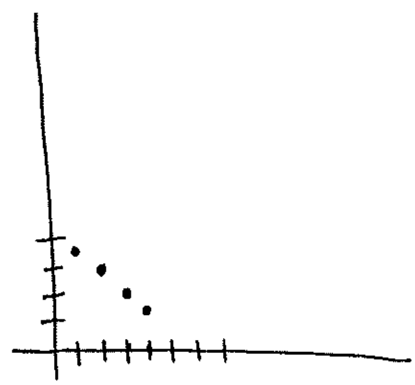
2. 8,5,2,-1,........



3. 3,7,11,15,........



4. $\frac{7}{2},3,\frac{5}{2},2,\frac{3}{2},\ldots\ldots.$



On the above graphs, if the Domain were all real numbers what would the equation of a sketch going through the points look like and what would the equation of the graph be.

1. line

$y = 2x + 1$

2. line

$y = {}^{-}3x + 11$

3. line

$y = 4x - 1$

4. line

$y = {}^{-}\frac{1}{2}x + 4$

As you can see the equation for this type of graph is linear when you keep adding the same amount. The traditional equation for dealing with this kind of equation is

$$y = mx + b$$

We now want to change this to Sequences notation so we switch the y to a_n and x to n to tell us we only want integer values for n.

$$a_n = (?)n + (?)$$

Together with your table partner please try and describe how you decide what the m and b change to in sequence notation.

m becomes the common difference, the number we add each time

b is the number we would have started at if there had been a 0'th term

6. Given the numbers from 1-100 please find the sum of all the numbers.

$1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100$

$$\frac{101 \cdot 100}{2} = 101 \cdot 50 = 5050$$

6. Given the numbers from 1-1000 please find the sum of all the numbers. (hint, try to find an easy way to add them up with patterns.

$1 + 2 + 3 + 4 + \cdots + 999 + 1000$

$$\frac{1001 \cdot 1000}{2} = 1001 \cdot 500 = 500{,}500$$

Activity 3: Geometric Sequences

Objective:

TSWBAT:

1. use the exponential relationship to write an explicit rule.
2. graph the exponential relationship of an geometric sequence.

Sequence of Activities & Prior Knowledge:

This activity can be used once students have explored the nature patterns in activity 1. The students would have the need to have the knowledge of sequence notation. The students would also need to be familiar with traditional graphing. The other prior knowledge needed is the standard form of exponential equation. The student would need to know the difference between the terms continuous and discrete or that will need to be explained to them before they work on the worksheet.

Student Work Concerns:

The students very quickly realized it was similar to the prior activity on arithmetic patterns and so it went well once they graphed the patterns and realized it was related to the exponential.

Teacher Notes:

Similar to the prior worksheet on arithmetic sequences, I started by using a graphing approach to relate the geometric sequence to an exponential function. In second year algebra texts they use $a_n = a_1 r^{n-1}$ rather than the preferred $a_n = a_0 r^n$. This way the traditional parent function in exponentials of $f(x) = ab^x$ is more directly related for students who need to see the direct relation between the initial value, a and the a_0 and the growth factor, b with the r. Similar to arithmetic, I use an a_0 that doesn't really exist but it conceptually directly relates to the exponential function that students have studied only a few months earlier.

Geometric Sequences and Series

Name___________________

Date ___________________

Graph the following sequences using the term position as the Domain and the values as Range.

1. 3,6,12,24,48,.....

2. 40,20,10,5,.........

3. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

4. 81,27,9,3,1,........

On the above graphs, if the Domain were all real numbers what would the equation of a sketch going through the points look like and what would the equation of the graph be.

1.

2.

3.

4.

As you can see the equation for this type of graph is linear when you keep adding the same amount. The traditional equation for dealing with this kind of equation is

$$f(x) = ab^x$$

We now want to change this to Sequences notation so we switch the f(x) to a_n and x to n to tell us we only want integer values for n.

$$a_n = (?)(?)^n$$

Together with your table partner please try and describe how you decide what the a and b change to in sequence notation.

a

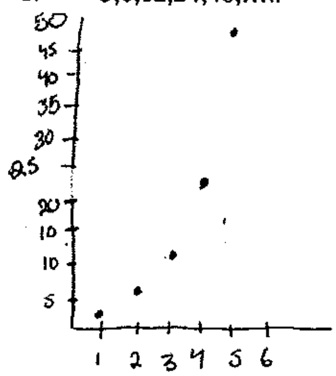
b

Geometric Sequences and Series

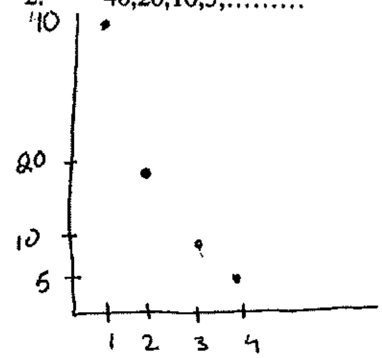
Name 3
Date

Graph the following sequences using the term position as the Domain and the values as Range.

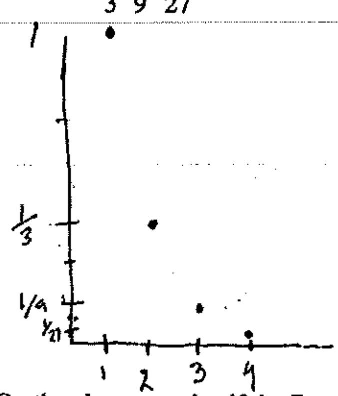
1. 3,6,12,24,48,.....



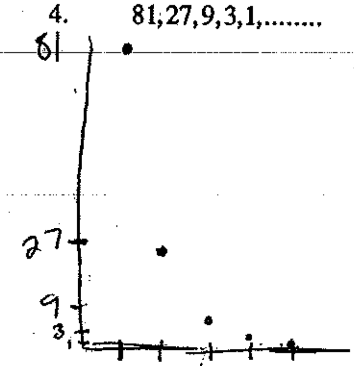
2. 40,20,10,5,.........



3. $1,\frac{1}{3},\frac{1}{9},\frac{1}{27},....$



4. 81,27,9,3,1,........



On the above graphs, if the Domain was all real numbers what would the equation of a sketch going through the points look like and what would the equation of the graph be.

1. $y=3\times 2^{(n-1)}$

2. $y=80\times \frac{1}{2}^{n}$

3. $y=\frac{1}{3^{(n-1)}}$

4. $y=81(\frac{1}{3}^{n-1})$

As you can see the equation for this type of graph is linear when you keep adding the same amount. The traditional equation for dealing with this kind of equation is

$$f(x) = ab^x$$

We now want to change this to Sequences notation so we switch the f(x) to a_n and x to n to tell us we only want integer values for n.

$$a_n = (?)(?)^n$$

Together with your table partner please try and describe how you decide what the m and b change to in sequence notation.

a A has to do with where the sequence starts ~~A is the starting point~~

b B is the factor by which the sequence changes for each n in the sequence.
If it is divided by 3 each time, B

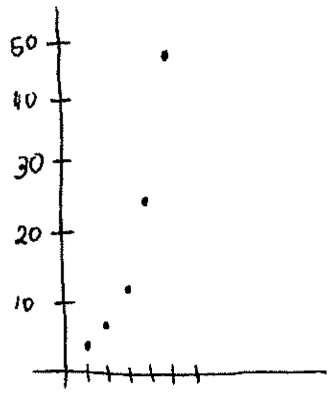
Geometric Sequences and Series

Name Key

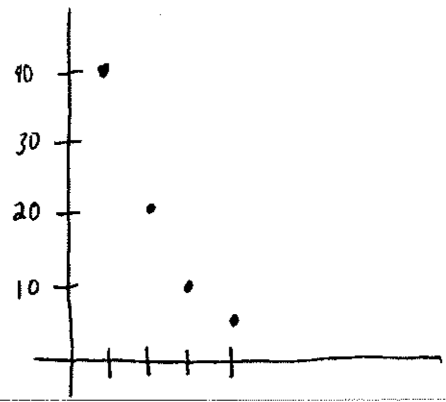
Date ______________

Graph the following sequences using the term position as the Domain and the values as Range.

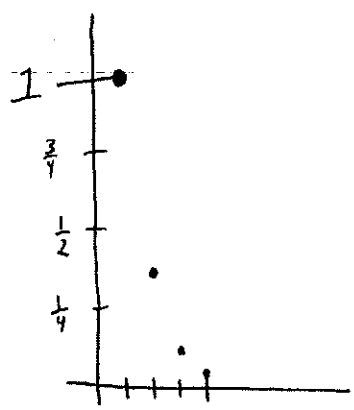
1. 3,6,12,24,48,.....



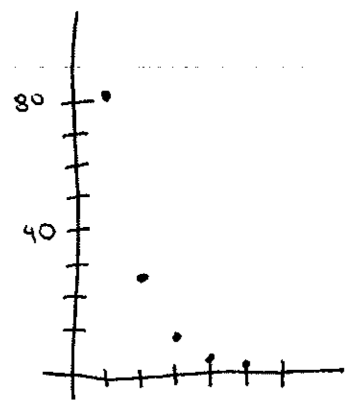
2. 40,20,10,5,.........



3. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$



4. 81,27,9,3,1,........



On the above graphs, if the Domain were all real numbers what would the equation of a sketch going through the points look like and what would the equation of the graph be.

1. exponetial growth

$y = \frac{3}{2}(2)^x$

2. exponetial decay

$y = 80(\frac{1}{2})^x$

3. exponetial decay

$y = 3(\frac{1}{3})^x$

4. exponetial decay

$y = 243(\frac{1}{3})^x$

As you can see the equation for this type of graph is linear when you keep adding the same amount. The traditional equation for dealing with this kind of equation is

$$f(x) = ab^x$$

We now want to change this to Sequences notation so we switch the f(x) to a_n and x to n to tell us we only want integer values for n.

$$a_n = (?)(?)^n$$

Together with your table partner please try and describe how you decide what the m and b change to in sequence notation.

a is like the y-intercept, what value would have been the 0'th term

b the common ration, what are you multiplying by each time. If it is greater than 1 it is growth and if it is between 0 and 1 it is decay

Activity 4: Practice of Sums of Series

Objective:

TSWBAT:

1. find the sums of arithmetic and geometric series.
2. discuss the other unique series after making educated guesses.

Sequence of Activities & Prior Knowledge:

The students would have to have finished the first three activities. They also would have had to practice sums of geometric series.

Student Work Concerns:

The three students did quite well with the geometric and the arithmetic series. The other series did a good job of confusing the students into arguments on if the series would have a sum or not. I tried in the directions to make sure they realized that there are certain sums we had looked at together and others they would have to make educated guesses about. After they had come to their own conclusions they asked me to explain the “weird” ones as they called them. I actually pulled out my history section of my Masters' paper and they had me go into detail explaining the proofs of the harmonic series and the reciprocals of the squares.

Teachers Notes:

This worksheet is to explore the idea of which series have sums and which ones don’t. I even added the harmonic series and the reciprocals of the squares as good discussion problems yet beyond the level of my high school students. This is a prime example of something that would work well with a class that can do well on self-directed assignments or exploratory work. The difficult part would be how do you deal with a class that refuses to do such a thing. This thought comes up because I have been teaching three algebra two classes this year and two of the three could handle the independence. This is mainly an issue I need to address in training a class well to think on their own.

Practice of Sums of Series

Name__________________
Date __________________

For the following Series, please state if you think they have a finite sum, infinite sum, or we are not sure which and why. Then if you feel it does have a sum, please try and find it. (hint: we have not tried all these types of Series so do your best you can)

1. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+......$

2. $2+4+8+.....+256+512$

3. $2+3+4+5+.......$

4. $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+.....$

5. $14+11+7+4+....-14+-17$

6. $\frac{1}{8}+\frac{1}{4}+\frac{1}{2}+1+2+4+8+....$

7. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...$

8. $125+25+5+1+\frac{1}{5}+\frac{1}{25}+.....$

9. $-10+-5+0+5+10+15$

10. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+....$

Practice of Sums of Series

Name Key

Date ________________

For the following Series, please state if you think they have a finite sum, infinite sum, or we are not sure which and why. Then if you feel it does have a sum, please try and find it. (hint: we have not tried all these types of Series so do your best you can)

1. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+......$

finite sum

1

2. $2+4+8+.....+256+512$

finite sum

1022

3. $2+3+4+5+.......$

infinite sum

4. $\frac{1}{1}+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+.....$

finite sum

5. $14+11+7+4+....-14+-17$

finite sum

-19

6. $\frac{1}{8}+\frac{1}{4}+\frac{1}{2}+1+2+4+8+....$

infinite sum

7. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...$

infinite sum

8. $125+25+5+1+\frac{1}{5}+\frac{1}{25}+.....$

finite sum

$125\cdot\left(\frac{1}{1-\frac{1}{5}}\right)$

156.25

9. $-10+-5+0+5+10+15$

finite sum

15

10. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+....$

finite sum

$\frac{\pi^2}{6}$

Activity 5: The Sierpinksi Triangle

Objective:

TSWBAT:

1. explain limiting values of repeating patterns.
2. take patterns and turn them into explicit rules for geometric and arithmetic patterns.

Sequence of Activities & Prior Knowledge:

This would be an activity to use after all the other objectives have been addressed in the work the class have used on series and sequence. I would look at these last two activities as ways to look at patterns, sums, and limits all at the same time. It will be a difficult activity for many Algebra 2 students in terms of their problem solving skills.

Student work concerns:

In retrospect, I needed to work on making sure the directions are more concise before I use this worksheet in the future. I proofed it over a couple of times yet it made a lots less sense looking back at it when we used it a few days later. The students very graciously asked for points of clarification, which I was more than happy to help provide. ~~The first area of clarification was how to actually create the different stages of the~~ triangle. An important question I neglected to put on here would the number of triangles and remaining area become as n goes to infinity. I proposed that question on the board with my students and they very quickly answered with appropriate answers of infinity ~~and zero. The corrections are made on the worksheet included.~~

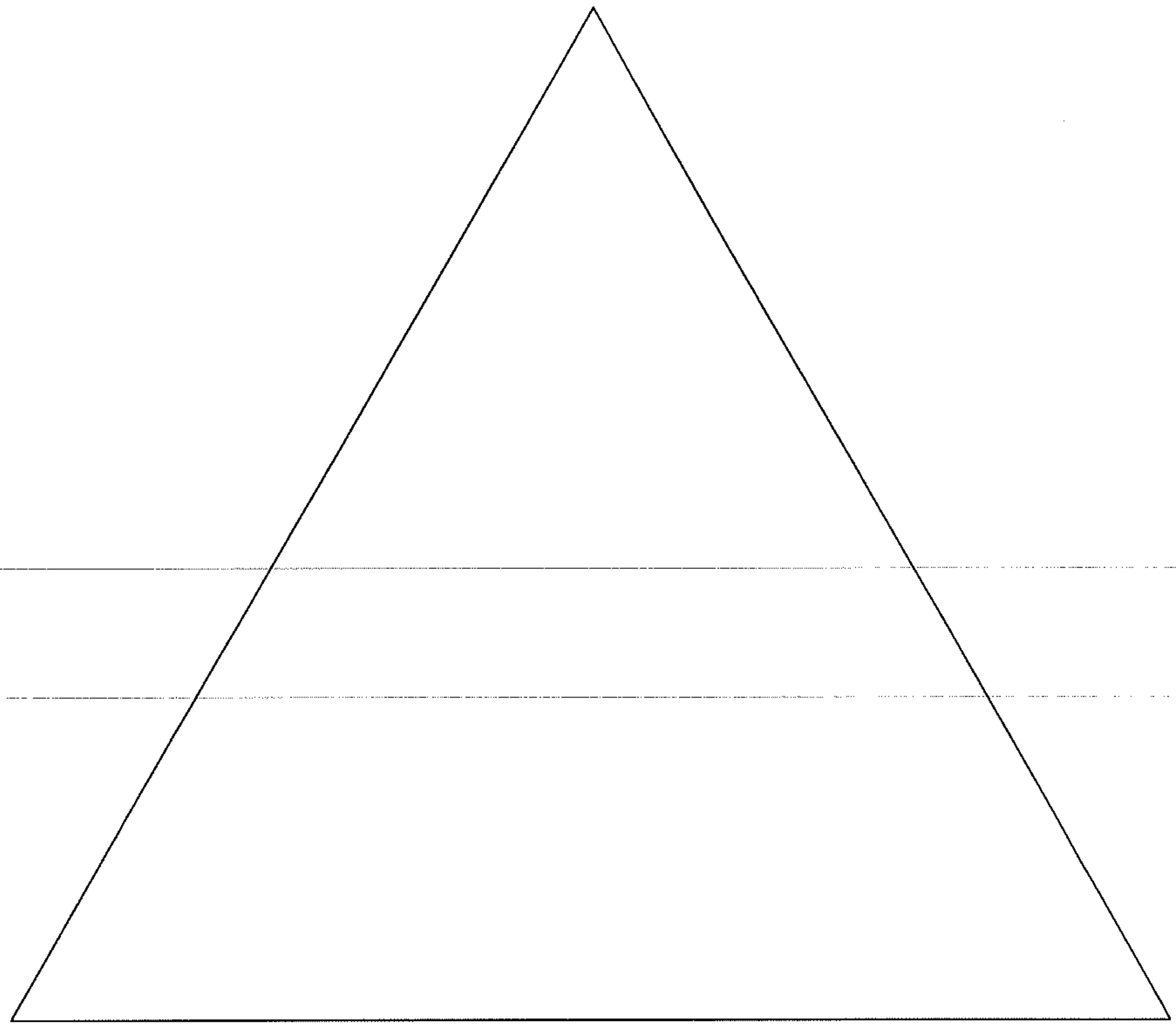
Teacher Notes:

I do really enjoy this problem as it lead in quite well to the snowflake problem and yet is good on its own. As a teacher you are looking at an example of a limit to find the remaining area. The students have not formal training of limits; they just need to look at the pattern and extend what they think it would be going to infinity. It is also the first time the students will have seen something like this where the number of triangle added goes to infinity but the area goes to zero. It is conceptually easier for the students to understand this one than the next snowflake problem so that is why we start with this one.

The Sierpinski Triangle

Name_________________

Date __________________



Take the equilateral triangle from above and find the midpoint of each side and connect them. Then shade in the inside triangle. This is figure one. Then take each of the three equilateral triangles that are not shaded and do the same process to each. Each level after shading is considered a stage of the Sierpinski Triangle. Try and draw up through the 3rd stage.

List the series of numbers that represent how many shaded triangles are added at each stage of the process of creating more triangles. Describe the pattern of numbers you listed. Write an expression to find a_n.

How many triangles would be additionally shaded at the 14th stage?

How many triangles would be shaded in after infinity stages?

What would the area of the non-shaded region be in each stage?

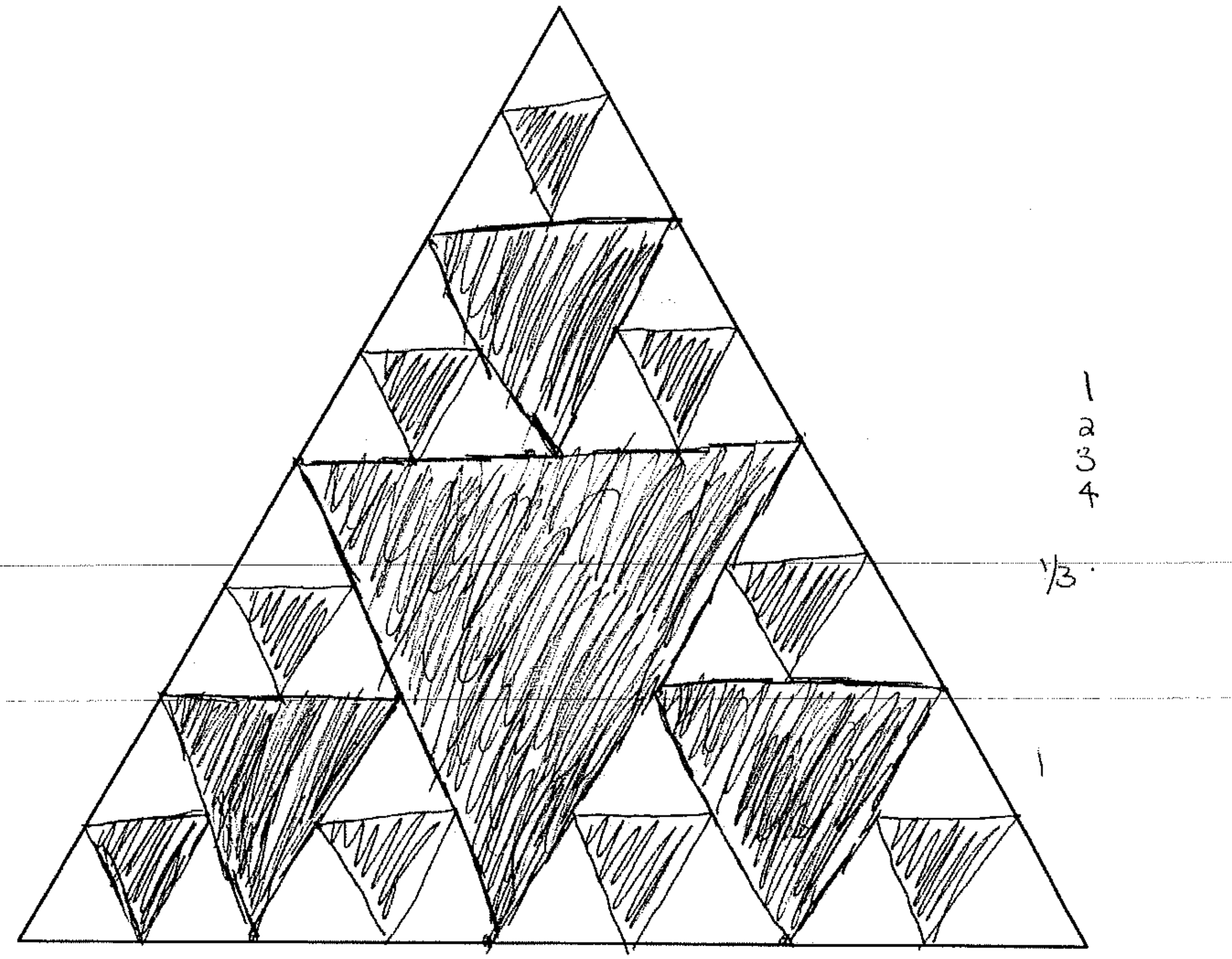
Write a rule for the remaining non-shaded region at the nth stage?

Find the remaining area of the original triangle at the 11th stage?

Find the remaining area of the original triangle after infinity stages?

The Sierpinski Triangle

Name Student # 2
Date 06/13/05



Take the equilateral triangle from above and find the midpoint of each side and connect them. Then shade in the inside triangle. Then take each of the three equilateral triangles on the outside and do the same process to each. Each level after shading is considered a stage of the Sierpinski Triangle. Try and draw up through the 3rd stage.

List the series of numbers that represent how many shaded triangles are added at each stage of the process of creating more triangles. Write an expression to find a_n.

$a_n = \frac{1}{3} \cdot 3^n$

How many triangles would be shaded in after the 14th stage?

$a_n = \frac{1}{3} \cdot 3^{14}$

$a_n = 1.6 \times 10^6$ triangles

What would the area of the non-shaded region be in each stage?

The non-shaded region would the total area of the triangle m the total area of shaded triangle region

3/4 of the region . . .

Write a rule for the remaining non-shaded region at the nth stage?

$a_n = \frac{3}{4}^n$ or $a_n = .75^n$

Find the remaining area of the original triangle at the 11th stage?

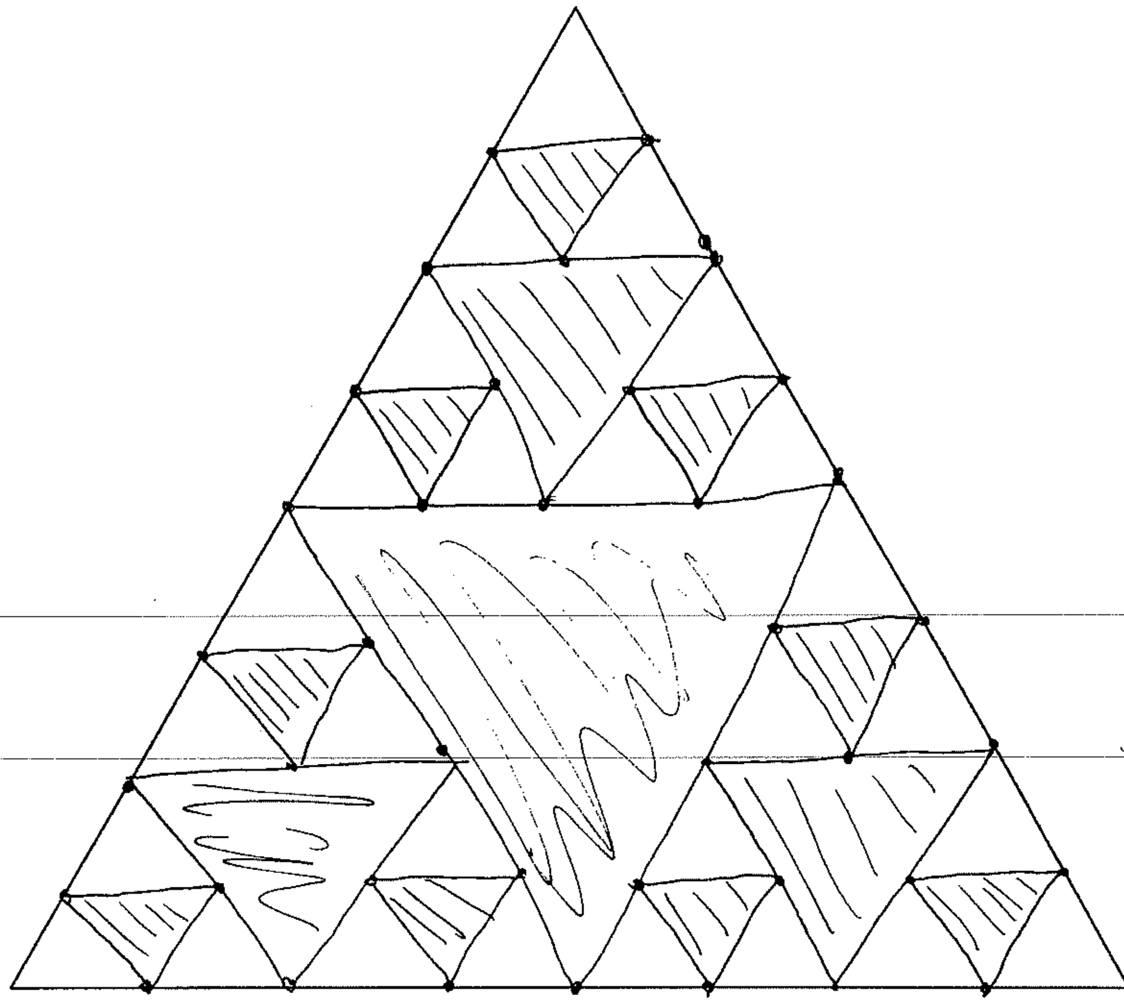
$a_n = \frac{3}{4}^{11}$

$a_n = .042235$

The Sierpinski Triangle

Name Key

Date ________________



Take the equilateral triangle from above and find the midpoint of each side and connect them. Then shade in the inside triangle. This is figure one. Then take each of the three equilateral triangles that are not shaded and do the same process to each. Each level after shading is considered a stage of the Sierpinski Triangle. Try and draw up through the 3rd stage.

List the series of numbers that represent how many shaded triangles are added at each stage of the process of creating more triangles. Describe the pattern of numbers you listed. Write an expression to find a_n.

1, 3, 9, 27, ...

$a_n = \frac{1}{3}(3)^n$

How many triangles would be shaded in ~~after~~ the 14[th] stage?

$a_n = 1{,}594{,}323$

How many triangles would be shaded in after infinity stages?

infinite, it just keeps going up

What would the area of the non-shaded region be in each stage?

$\frac{3}{4}$, $\frac{3}{4} \cdot \frac{3}{4}$, $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$

Write a rule for the remaining non-shaded region at the nth stage?

$a_n = \left(\frac{3}{4}\right)^n$

Find the remaining area of the original triangle at the 11[th] stage?

$a_n = \frac{177{,}147}{4{,}194{,}304}$

Find the remaining area of the original triangle after infinity stages?

zero, it gets smaller forever

$\lim_{n \to \infty} \left(\frac{3}{4}\right)^n = 0$

Activity 6: The Snowflake Problem

Objective:

TSWBAT:

1. explain limiting values of repeating patterns.
2. take patterns and turn them into explicit rules for geometric and arithmetic patterns.
3. take geometric patterns and find their sum.

Sequence of Activities & Prior Knowledge:

The prior knowledge the student would need for this activity is large. They would need to use sequence notation, describe patterns, explain limiting values, write exponential function equations, find geometric sums. This would be the last activity that I would use in my Algebra 2 class to reinforce the ideas of limits, sums, and patterns.

Student Work Concerns:

Finding the area and perimeter of the snowflake problem proved to be quite a challenge for my students which leads me to believe I must find a way to help guide the students more in this activity. By working on the wording so we can look at the area of the first addition, 2nd, etc separately from the initial area it will hopefully help us see the geometric series more easily.

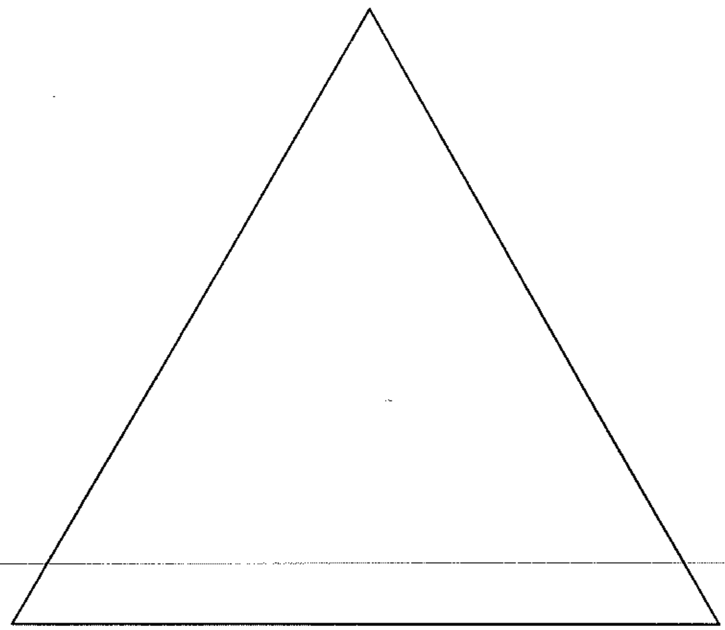
Teacher Notes:

The three students who helped me by working on these activities were the three highest grades in my Advanced Placement statistics class this last year. They worked together quite diligently and work on this for a good hour and a half and then sought my guidance to try and help them towards the right direction as they were getting frustrated. They were confident that the perimeter was infinite and the area was finite but couldn't find the value of the area. Due to this I probably not use this worksheet next year in my Algebra two class but will seriously edit the worksheet to try to make it more useful for me in the classroom. My goal in making this activity was to have a difficult activity that would try to tie together the ideas of limits, sums, and patterns at the same time. If it gave this much difficulty to my three Advanced Placement Statistics students my Algebra 2 class wouldn't be able to handle it without tremendous assistance. This is difficult for me as I like activities that students can handle with minor input from me.

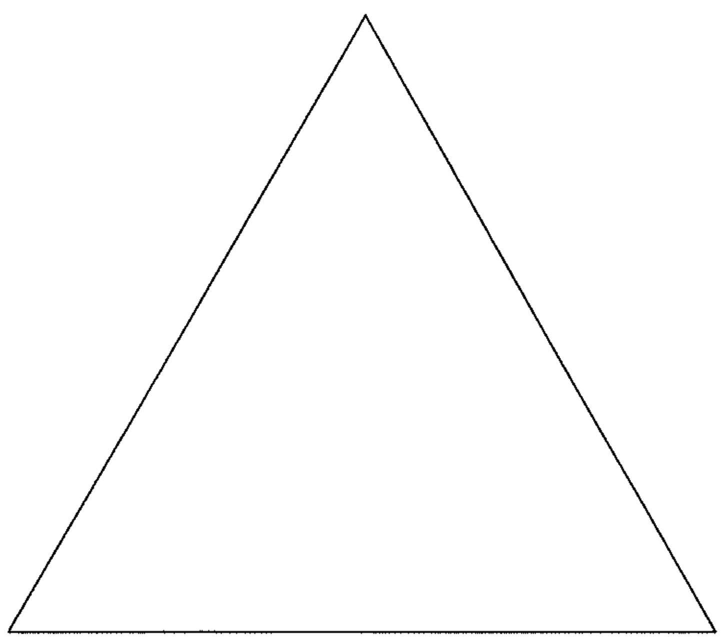
The Snowflake Problem

Name__________________
Date__________________

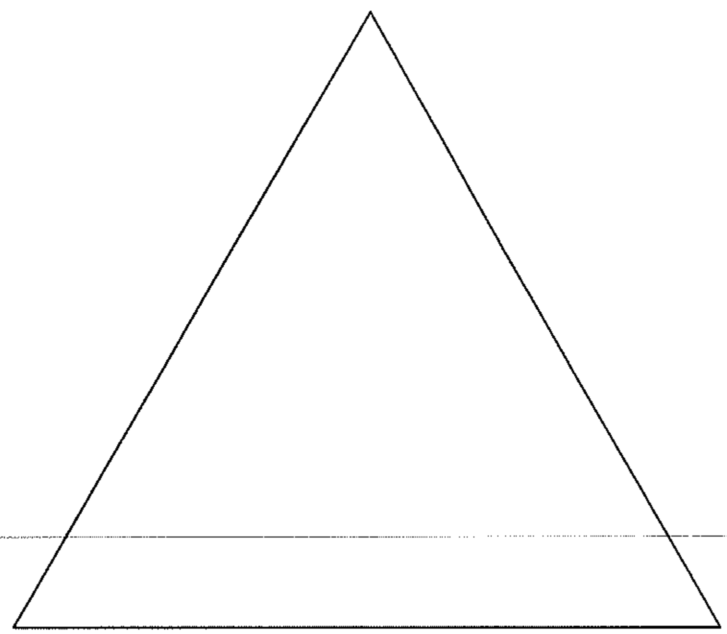
The following triangle has area A and perimeter P.



Take out the middle one third of each side of the triangle and use the piece you remove as the base of new equilateral triangle pointing outward.



Do the same thing to this triangle as with the previous one. Now do it again on every one of the twelve sides.



What is the area of the first drawing, the additional area of the second, the third?

What is the pattern in the numbers for future stages of the same process?

If this process were taken to infinity, what would be the total area?

What is the perimeter of the first drawing, the perimeter of the second drawing, the perimeter of the third drawing?

What is the pattern in the numbers for future stages of the same process?

Write a formula to find the perimeter in the nth stage.

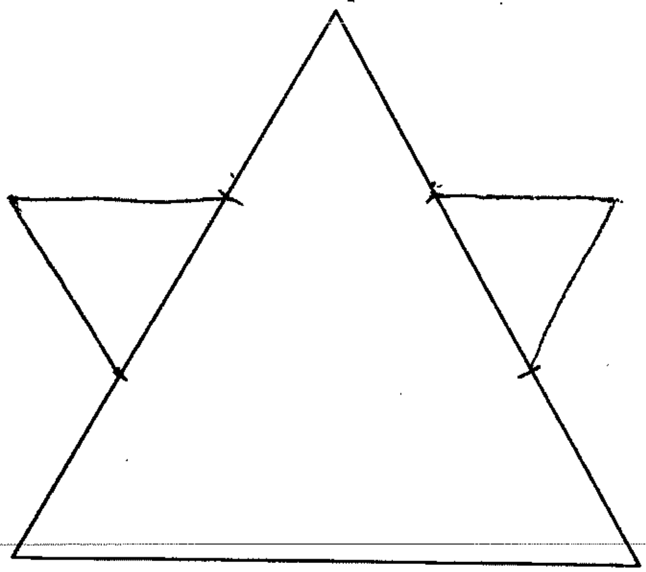
If this process were taken to infinity, what would be the total perimeter?

Snowflake Problem

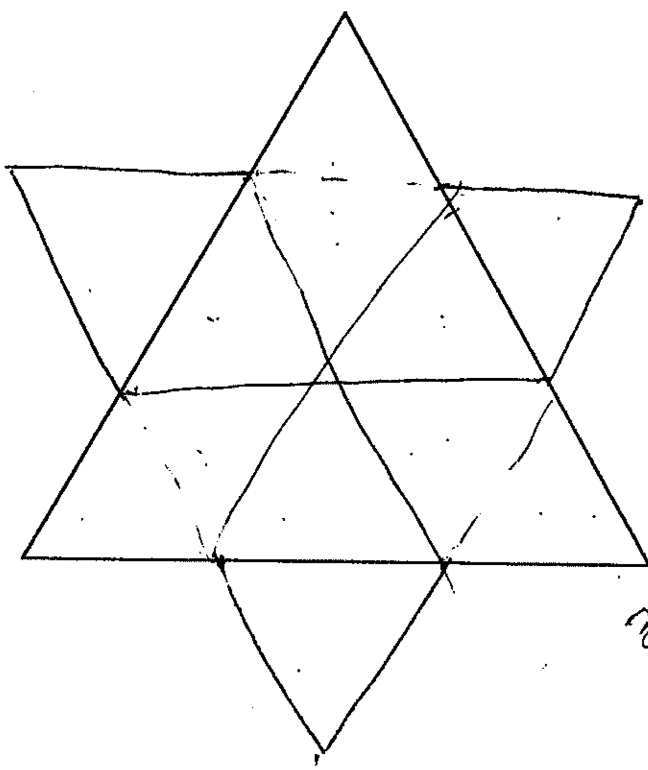
Name 3

Date

The following triangle has area A and perimeter P.



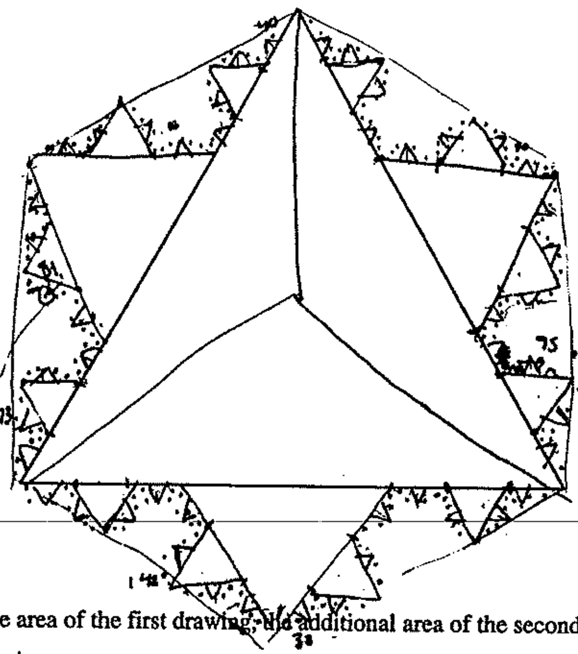
Take out the middle one third of each side of the triangle and use the piece you remove as the base of new equilateral triangle pointing outward.



3(16 93   48 / 729   8 / 9 72°

3

Do the same thing to this triangle as with the previous one. Now do it again on every one of the twelve sides.



What is the area of the first drawing, the additional area of the second, the third?

$a_1 = 1$

$a_2 = \frac{1}{3}$

$a_3 = \frac{12}{81} = \frac{4}{27}$

What is the pattern in the numbers for future stages of the same process?

$a_n = \frac{\quad}{\frac{1}{9}9^n}$

Write a formula to find the area added in the nth stage.

$a_1 = \frac{1}{3}$ $\quad a_n = \frac{3(4^{n-1})}{\frac{1}{9} \times 9^n}$ unless $n = 1$

3.

If this process were taken to infinity, what would be the total area?

a finite number

8/5

What is the perimeter of the first drawing, the perimeter of the second drawing, the perimeter of the third drawing?

$a_n = \left(\frac{4}{3}\right)^n$

What is the pattern in the numbers for future stages of the same process?

gets bigger

Write a formula to find the perimeter in the nth stage.

$a_n = \left(\frac{4}{3}\right)^n$

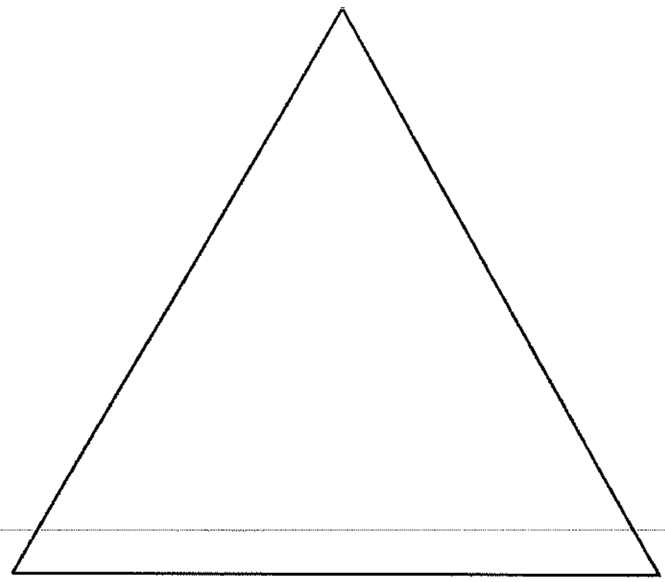
If this process were taken to infinity, what would be the total perimeter?

∞

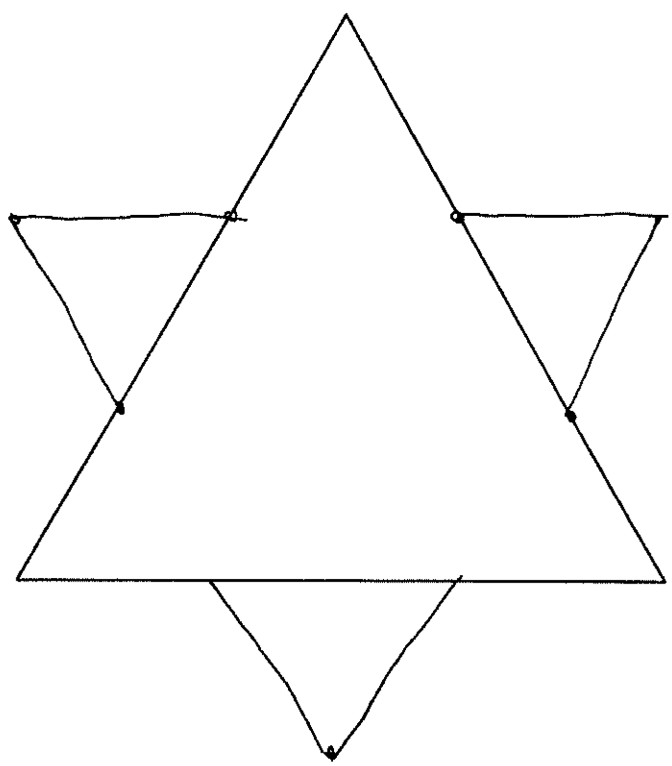
The Snowflake Problem

Name Key
Date ______

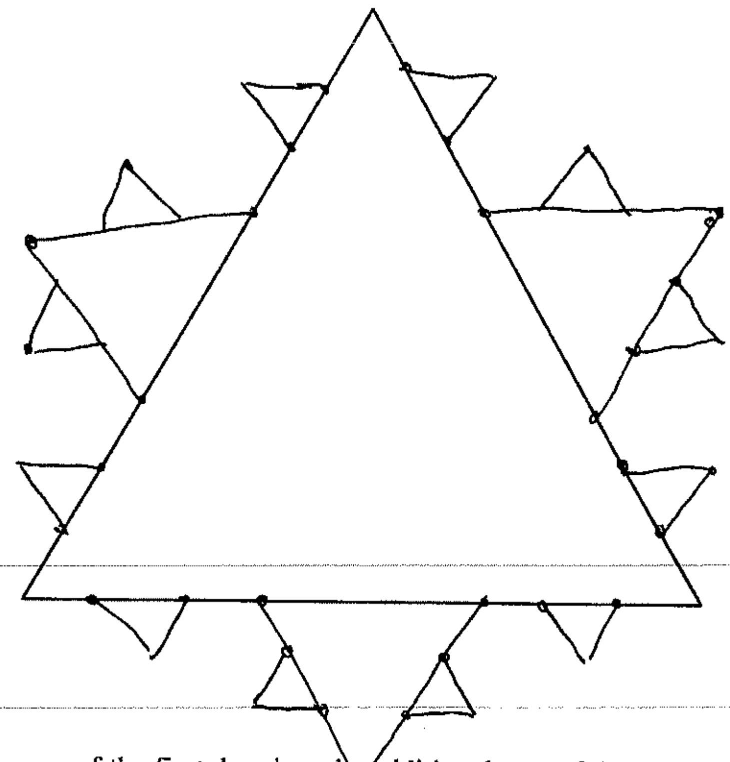
The following triangle has area A and perimeter P.



Take out the middle one third of each side of the triangle and use the piece you remove as the base of new equilateral triangle pointing outward.



Do the same thing to this triangle as with the previous one. Now do it again on every one of the twelve sides.



$$\frac{12}{27} = \frac{4}{9}$$

What is the area of the first drawing, the additional area of the second, the third?

$$A\left[1 + \frac{1}{3} + \frac{4}{27} + \frac{16}{243} + \cdots\cdots\right]$$

What is the pattern in the numbers for future stages of the same process?

add $\frac{4}{9}$ area of the previous addition

If this process were taken to infinity, what would be the total area?

$$A + A\left[\frac{1}{3} + \frac{4}{27} + \frac{16}{243} + \cdots\cdots\right]$$

$$A + A\left[\frac{1}{3} \cdot \frac{1}{1-\frac{4}{9}}\right] = A + A\left[\frac{1}{3} \cdot \frac{9}{5}\right]$$

$$\frac{8}{5}A$$

What is the perimeter of the first drawing, the perimeter of the second drawing, the perimeter of the third drawing?

P , $\frac{4}{3}P$, $\frac{4}{3} \cdot \frac{4}{3} P$

What is the pattern in the numbers for future stages of the same process?

times by $\frac{4}{3}$ the last perimeter

Write a formula to find the perimeter in the nth stage.

$a_n = P\left(\frac{4}{3}\right)^{n-1}$ or $a_n = \frac{3}{4}P\left(\frac{4}{3}\right)^n$

If this process were taken to infinity, what would be the total perimeter?

$\lim_{n \to \infty} P\left(\frac{4}{3}\right)^{n-1}$

$P \lim_{n \to \infty} \left(\frac{4}{3}\right)^{n-1}$

Infinity

Curriculum Reflection

Overall I am happy with the work I have put together on series and sequences for high school students. The curriculum that comes with the high school textbooks on the subject are often lacking in motivation and explanation. By incorporating historical anecdotes and being able to share conceptions about math, I have been able to motivate the students and show them an appreciation for the subject matter. I believe the materials I have created are going to be extremely helpful for the Math department at Centennial High School. The tougher conceptual activities will have to be used with caution, depending on the students you have in your high school class. Upon reflection I should have proofread better at the beginning. I am thankful for the patience of my students who helped me review the activities. I also want to personally thank my fellow Algebra 2 teachers at Centennial who helped me come up with some of the ideas and pointed me in the right direction.

Reflections and Acknowledgments

The first area I would like to discuss is the long term process of working on my master's project through the last year. I feel like I worked well in spurts and not in others as my life was too busy to spend lots of time working on this MST project every week. It is difficult to work on a project like that throughout the year. In spite of that I am proud that every few months I set out a new goal and then proceeded to get done with that phase of my project.

I do feel I have learned a lot to help me progress as a better math teacher when it comes to teaching Algebra 2 Series and Sequence curriculum. I would like to have this kind of time to study each of the topics that I teach as a high school teacher. Given the time constraints of teaching that already take over 60 hours a week, I am going to have a goal during my summertime to read three books related to a particular math topic I am teaching. When it comes to literature, I feel I can always improve as a math teacher. I always say that if I ever feel like I am a perfect teacher I will resign and go into another line of work.

In terms of acknowledgments I would like to thank my committee of Mike, Karen, and John. Their guidance has helped lead me to the project that I am quite proud of today. I would like to also thank my Mom and Dad as they have given me the support of helping me grow up to be the young man that I am today. When different parts of my project were stressing me out they would talk to me and help me relax. I also would like to thank Kristen Teel, a college friend who helped me proofread and organize my paper. I thank her for her patience. The last group of people I would like to thank is the Math

Department at Centennial High School who have helped me develop my math philosophy that I have today.

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http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Euler_Leonhard.html (JJ O'Connor and E F Robertson)

http://www-history.mcs.st-ndrews.ac.uk/history/Biographies/Sierpinski.html
(JJ O'Connor and E F Robertson)

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People in Series and Sequences:

As with any part of mathematics there are a few people who have had a great influence on the development in Series and Sequences summing. I would like to specifically like to focus on a few Mathematicians who found sums in Series and Sequences. I have included this as an appendix as it was valuable to the start of my research but not vital to the entire project.

Jacob (Jacques) Bernoulli (1654-1705)

Jacob Bernoulli's father inherited a spice business and was quite important member of the town council in Basel. He studied philosophy and theology due to his family's influence. He received his Masters degree in philosophy in 1671 and a licentiate (advanced degree) in theology in 1676. During the same time he was studying theology and philosophy he studied math and astronomy against his families wishes. Over the next few years he worked with Descartes, Malebranche, Boyle, and Hooke on his math and astronomy. After this time he started many written conversations with mathematicians he carried on for many years.

He returned to Switzerland and taught at the University of Basel from 1683. He had an opportunity to start working for the church but declined, as he would rather work on math and theoretical physics. Jacob married in 1684 and had two children who choose not to study math, which is a surprising giving the rest of the family, seemed to study math.

Then one of the most important things Jacob did was started to teach his brother Johann math while his father had him study medicine. Jacob then became a professor of

Math in Basel in 1687. He and his brother studied the papers of Leibniz's on calculus. This led to a public split of Johann and Jacob about whom was the best mathematician. Due to Jacob being the head Mathematician in Basel it meant that Johann had to move to Holland to get a post in 1695.

Jacob Bernoulli's first major contribution was on Logic, Algebra, Geometry, and Probability. The geometry result gave a construction to divide any triangle into four equal parts with two perpendicular lines. Then later in 1689 Jacob published work on infinite series and his law of large numbers in probability theory. This interpretation of relative frequency led to our modern idea in probability of the law of large numbers. Jacob Bernoulli published five treatises on infinite series between 1692 and 1704. He produced something that he thought was new that Mengoli had discovered 40 years earlier. He showed that $\sum \frac{1}{n}$ diverges and that $\sum \frac{1}{n^2}$ converged to a finite number less than 2. Euler found this sum later in 1737.

One of the most famous things you think of when you hear Bernoulli is "the Bernoulli equation" This was Jacob's work on solving differential equation.

$$y' = p(x)y + q(x)y^n$$

He worked quite a bit on logarithmic spiral and epicycloids. He had a book on probability that was published eight years after his death that wasn't complete but still contributed tremendously to the theory of probability (JJ O'Conner).

Johann Bernoulli (1667-1748)

Jacob's brother Johann is one of the next most influence mathematicians on this subject. Being the youngest of ten siblings his big brother had an early influence on him. As Johann's parents tried to get him to run the family spice business he wanted to go and study something besides business. He went to the University of Basel were his brother was lecturing on experimental physics and studied medicine. His big brother started tutoring him on the side so he could try and learn math instead of medicine. He and his brother eventually studied Leibniz's Calculus together and became "equal" math students.

In 1691 Johann went to Paris and met with de L'Hopital and had some deep mathematical conversations. Johann actually spent serious time tutoring him in the ideas behind calculus. Even after he returned to Basel he continued to correspond with L'Hopital and he was paid half of a professor's salary by L'Hopital to continue to help him learn calculus. Johann was actually disappointed when L'Hopital published a book based on Johann's methods and the only mention was in the preface which said, "And then I am obliged to the gentlemen Bernoulli for their many bright ideas; particularly to the younger Mr. Bernoulli who is now a professor in Groningen. Many years later some work was recovered to show that L'Hopital's rule was really discovered by Johann. The work L'Hopital did make it more precise.

Johann started regular communication with Leibniz on math while finishing his doctorate in math. During the next few years he worked on and off with his brother on math but they could not publish together as they had developed a rivalry. This lead to the most fruitful time of his mathematical discovery by working on integration as the reverse of derivatives, summed series, and discovered additional theorems in trigonometric and hyperbolic functions using the differential equations that they satisfy. Johann Bernoulli worked as a professor in Groningen for ten years.

A former student, Petrus Venhuysen accused him of opposing the Calvinist faith. Johann said,

"… I would not have minded so much if [Venhuysen] had not been one of the worst students, an utter ignoramus, not known, respected, or believed by any man of learning, and he is certainly not in a position to blacken an honest man's name, let alone a professor known throughout the learned world….all my life I have professed my Reformed Christian belief, which I still do… he would have me pass for an unorthodox believer, a very heretic; indeed very wickedly he seeks to make me an abomination to the world, and to expose me to the vengeance of both the powers that be and the common people…"

While Johann held the chair position in Groningen, he was competing with his brother in an interesting mathematical rivalry. In 1705 he went back to Basel with his wife to visit her sick father and Johann convinced the university to let Johann take over the chair of mathematics after his brothers passing. His father in law actually lived for 3 years before passing so the family got to be in close proximity.

Johann supported Leibniz calculus and Descartes vortex theory over Newton's theories of Calculus and Gravity. He was correct in Leibniz calculus was more exact yet was wrong on Descartes vortex theory. This actually led to Newton's theory of gravity not being widely accepted until later. His competitive nature lead to him competing with his own son Daniel about when they dated their work they published in the 1730's (JJ O'Connor).

Leonhard Euler (1707-1783)

Leonhard Euler was born in Basel where his father had actually lived with Johann Bernoulli in Jacob Bernoulli's house for a time. His family moved to nearby Reihen. Even though Euler's father was a minister he had an elementary training in math to teach his son growing up. Euler was sent to Basel to go to a poor school that lacked mathematical training. Johann Bernoulli learned of Euler's enthusiasm for the subject. He soon would be reading math books on a weekly basic and then asking Johann questions every Sunday afternoon. At the age of 19, Euler won a prize from the French Academy for where to put the mast of a ship before he had ever seen an ocean going vessel (Dunham, p. 208).

Euler completed his Masters in philosophy having compared the ideas of Descartes and Newton. He didn't want to purse this so with the help of Johann he convinced his father to let him study mathematics. Then at the age of 19 he accepted a physiology post at St. Petersburg. Through the requests of Daniel Bernoulli and Jakob Hermann Euler was appointed to the mathematical-physical division of the academy instead. From the years of 1727-1730 he also served in the Russian Navy to support his work at the Academy. In 1730 he became the Professor of Physics so he could become a full member of the academy and leave his Navy post. When Daniel Bernoulli left to go back to Basel Euler was appointed to be in charge of the math department.

This helped Euler's monetary situation so he could marry Katharina Gsell and have 13 children. Only 5 children survived infancy but Euler claimed he made some of his greatest mathematical discoveries while holding a baby. Euler had health problem that made him lose his eyesight over time. Due to political pressures in Russia, Euler accepted a post in Berlin were he worked for St. Petersburg Academy by mail. That way his still received some of his salary from St. Petersburg while working in Berlin. During Euler's time in Berlin he published around 380 articles. Due to a conflict with who was in charge he left his post in Berlin and went back to Russia (JJ O'Connor).

When Euler passed away he had published 560 papers and books. He left behind many manuscripts behind that were published for the next 47 years. Gustav Enestrom's research helped expand the list 856, plus 31 things that were published by his eldest son which were written under his father's direct instruction (Struik, p. 120). Euler had assistants who would work for him and Euler granted them credit in his published works.

Euler is the person who we owe for many of the notation we use today. Things like the notation of $f(x)$, i, e, and π we owe to Euler. One of the most important discoveries in series was to find $\sum \frac{1}{n^x}$ when x is whole numbers. This problem led to other being solved using the fact that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. In his work Euler came up with the formula used in Analysis $e^{ix} = \cos x + i \sin x$ (JJ O'Connor).

Even with the greatest achievements that Euler made in the area of infinite series he still had some major issues in his logic. Euler concluded that –1 is larger than infinity. He also argued that infinity separates positive and negative numbers just as zero does. So Euler was the most amazing mathematician of his time, yet still had ideas that over the past century have been shown to be incorrect (Kline, p.144).

Waclaw Sierpinski (1882-1969)

Sierpinski grew up in Warsaw during a time when the Russian Empire was in control, which made it difficult to get the schooling his family wanted for him. Sierpinski graduated from the University of Warsaw in 1904. He taught at a girls' school, and when the teachers at the school went on strike he decided to go to Krakov to study for his Doctorate. He received his Doctorate in 1907 and started working at the University of Lvov.

In 1907, he became interested in Set Theory. Then he gave the first lecture course devoted to Set Theory in 1909. During World War I, he ended up stuck in Moscow working with Luzin on analytic sets. In 1919 he returned to the university in Warsaw to work for the rest of his career. He worked mainly on Set Theory and Functions of Real Variable.

He was highly involved in the administration of the societies of math and science in Poland. He was the Chairman of the Polish Mathematical Society. He ran into difficulty in World War II, as the Nazis destroyed his house and papers. He secretly sent his math work to Italy to get published. According to Sierpinski, over half the mathematicians who had lectured in Poland were killed. The Nazis destroyed by fire the Warsaw University Library during the war.

According to Sierpinski's student Rotkiewicz, "Sierpinski had exceptionally good health and a cheerful nature. … He could work under any conditions. … He was the greatest and most productive of Polish mathematicians." (JJ O'Conner and EF Robertson) Sierpinski continued during his last decade of life to give lectures, even though he was already retired. He passed away in 1969 at the age of 87.